Introduction to

QR

Quantitative Reasoning

Course Notes for Math100

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What is Quantitative Reasoning?

Maybe the question should read: How is Quantitative Reasoning different from Mathematics?

Context is the answer. As mathematics curricula have evolved over the years, emphasis has been placed on the techniques and rules of algebraic manipulation. Much of the context and application have been stripped away to make the courses more compact, allowing them to fit into the crowded life of today’s student.

As a result, much of the motivation to learn mathematical principles has been stripped away with these applications. Only a minority of students are recognizing the value of this education, while many students forced to enroll in mathematics courses learn only what is needed to pass the course and forget the information learned soon afterward. Worse is the fact that those without quantitative reasoning skills are more easily manipulated by those with these skills, prompting book titles like Darrell Huff’s How to Lie with Statistics. Advertisers and politicians have much to gain by convincing large quantities of people to buy a product or support a policy backed by unsound quantitative reasoning.

Today’s mathematics student may learn the rules of algebra and know how to find the equation of a line, but that same student may have little idea what this knowledge can accomplish. Other than the fact this line equation may be an answer to a test question, how is a skill like finding line equations going to be useful for a student?

It is not surprising that a lack of application would turn away all but the mathematically devoted student. There is little motivation to learn knowledge that will not serve some use in the future. Emanuel Swedenborg summarizes this philosophy throughout his written works.

“It is from use, by way of use, and according to use, that life is imparted by the Lord. And that which is without use can have no life in it, for that which is without use is cast aside.”

Arcana Coelestia, no. 503

“Love and wisdom without use are merely abstract ways of thinking, which also after a brief stay in the mind, pass away like winds. But these two are gathered together in use, there becoming one, what is called reality.”

Married Love, no. 183.3

In this course, you will not only review the basics of algebra and probability, which will serve you should you choose to take other courses in mathematics, but you will also see how these principles can be applied to address quantitative issues you may encounter in other courses or in life. How do I compare gas prices in United States dollars per gallon against Canadian dollars per liter? Is it a good investment to buy that extended warranty? Has the price of food increased or decreased over the last ten years when compared to inflation? Is it really that unusual to hit five red traffic lights in a row on my way home? What exactly is my four-latte-a-week habit costing me over an entire year?
What is Discrete Mathematics?

One can categorize mathematics applications into two categories: Continuous and Discrete.

Continuous applications work with quantities that appear to be infinitely divisible. How long does it take to get from point A to point B? Between any two possible answers to this question there are an infinite number of possible answers in between, just as there are an infinite number of fractions between the values of zero and one.

Discrete applications work with quantities that come in whole numbers. How many ways are there to roll a number of dice? An answer to this question cannot be “nine and a half.” There may be nine or there may be ten, but an answer cannot take a value between these numbers.

Continuous applications exploded with the discovery of calculus. Both Isaac Newton and Gottfried Leibniz independently developed this field in the late 17th century, which had many applications to physics and other sciences. Calculus is part of the standard college curriculum for most scientific and engineering fields, and many high schools offer some form of calculus to advanced or honors students.

While continuous mathematics was gaining popularity, mathematical concepts of a discrete nature were also being developed, beginning in the 17th century with the work of Blaise Pascal and his famous triangle. Without the obvious applications that calculus offered, discrete mathematics was not given as much attention. The study of combinations and probability interested a few devout fans, but there didn’t seem to be any important reason to learn this style of mathematics.

This all changed once computing power was introduced in the late 20th century. Computers could make calculations thousands of times faster than any person (now millions of times faster), and problems which used to have appeal only to those interested in theory now could be solved in practice. Optimization problems of a discrete nature could be solved, like “What is the best route for my delivery trucks?” or “How should I schedule my airline crews to minimize wasted travel?” A whole area of the new field now called Computer Science studies the structure and complexity of the algorithms used to solve these types of problems.

Continuous mathematics has about a 300 year head start, so discrete mathematics has yet to catch up in many mathematics curricula. It is hard to convince the world that something done a certain way for many years should be changed, especially when there is nothing wrong with studying calculus. But just as many different personalities take to many different subjects, a student talented in discrete mathematics might never discover this talent if he or she believes that all higher mathematics relates only to calculus.

I will note that not all scholars of Newton’s time ignored the importance of discrete structures. Emanuel Swedenborg, in his work *Divine Love and Wisdom*, explains the concepts of discrete versus continuous degrees by comparing discrete degrees to layers of the atmosphere, each individually distinct, and comparing continuous degrees to gradations of light to shade, or of heat to cold. Swedenborg punctuates the importance of understanding both concepts when examining the theological ideas outlined in his works.

“Without a concept of [continuous and discrete] degrees, one can know nothing of the difference between the three heavens, nor of the difference between the love and wisdom of the angels in them, nor of the difference between the warmth and light that they possess, nor of the difference between the atmospheres which surround and envelop them.”

*Divine Love and Wisdom*, no. 185
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0 Preliminaries

0.1 Numbers and Operators

Understanding the basics of manipulating mathematical expressions is at the heart of almost every mathematical problem. The building blocks for these expressions are numbers, operators, and variables.

Numbers

You should know about numbers before attempting to read this textbook, but you may not be aware that numbers are divided into categories. Some of these terms will be used later in this book, and if you take other courses with mathematical content, you will likely see these terms again.

Natural numbers, also called counting numbers, are positive whole numbers, used for... well... counting. Examples: 1, 2, 3, 26, 148, and 3,000,000 just to name a few. However, numbers that are not natural numbers are not called unnatural numbers.

Integers include all whole numbers, including natural numbers, zero, and negative whole numbers. Examples: 5, 782, 0, −6, and −378. Numbers that are not integers are sometimes called non-integer or fractional numbers.

Rational numbers are numbers that can be represented by a ratio of integers. This includes integers since any integer divided by one creates a ratio equal to that integer. Examples: 4 (since 4 = 4/1), 7/15, −6/28, and 1,999,999/2,000,000. Any finite decimal can be shown to be a rational number by choosing the correct power of ten for the bottom of the fraction. For example: 3.4 = 34/10 and 1.618034 = 1,618,034/1,000,000. Numbers that are not rational numbers are often called irrational. Proving that a number is irrational can be very difficult. In chapter seven, there is a proof that \(\sqrt{2}\) is irrational. Another irrational number, \(\pi\), which represents the length of the circumference of a circle with diameter one, was not proved to be irrational until 1770 when Johann Lambert discovered a proof.

The set of real numbers is the collection of both rational and irrational numbers. This set of numbers is all you need to do almost every mathematical application there is. Examples: 8, −12, \(17/5\), and \(\sqrt{2}\). However, numbers that are not real numbers are not called unreal numbers. Sometimes they may be called non-real.

The last group of numbers identified here are complex numbers, and applications of these numbers are rare. This book will not be using them, but you may see them in other mathematics courses. The simplest non-real complex number is \(\sqrt{-1}\), labeled \(i\). All other complex numbers are found by taking real multiples of \(i\) and adding them to other real numbers, like 4 + 12\(i\) or 8.99 − 2.3\(i\).
Operators

Operators are symbols describing ways to combine numbers, the simplest of which are addition (+), subtraction (−), multiplication (×), and division (÷). Exponentiation, which is the raising of numbers to powers, is not represented by a symbol, but by writing the exponent in a superscript font, like $4^2 = 4 \times 4$ or $3^3 = 3 \times 3 \times 3 \times 3$. Numbers raised to the second power are said to be squared, and numbers raised to the third power are often called cubed. The root operator uses $\sqrt{}$ for its symbol, and it is calculated by finding the number whose square is equal to its argument. For example: $\sqrt{9} = 3$ since $3^2 = 9$ and $\sqrt{64} = 8$ since $8^2 = 64$. It should be noted that $(-3)^2$ is also 9. In the case where we want to indicate that both 3 and $-3$ are answers of interest, we would use the notation, $\pm \sqrt{9}$.

Unlike reading, operators are not always evaluated from left to right. Operators should be evaluated in a certain order depending on their level. Operators on the same level are then evaluated from left to right. The first level includes powers and roots, the next level is multiplication and division, and finally, the last level is addition and subtraction, sometimes called sums and differences. Parentheses can be used to override this order by placing priority on the operations inside the parentheses. Some symbols, such as the root symbol or the bar for division act similarly to parentheses.

**Order of operations:**

1] Expressions in parentheses: ( )
2] Powers and roots: $a^x$, $\sqrt{x}$
3] Multiplication and division: $\times$, $\div$
4] Addition and subtraction: $+$, $-$

**Question 1:** Evaluate: $4 + 3 \times 5^2$.

**Answer 1:** There is nothing in parentheses, so the next level is to calculate roots and powers: $5^2 = 25$, so the problem is now $4 + 3 \times 25$. The next level is multiplication: $3 \times 25$ is 75, so now we have $4 + 75$, which is 79.

Note that some calculators may not observe the order of operations, and simply do the calculations in the order you enter them. If you type in this sequence into your calculator: $4 + 3 \times 5^2$, you are likely to get 1225 instead of 79. This is because $(4 + 3) \times 5^2 = 7 \times 5^2 = 35^2 = 1225$.

**Question 2:** Evaluate: $6 \times (4 - 1)^3$.

**Answer 2:** First calculate the quantity in parentheses, which makes the problem $6 \times 3^3$. Then calculate the power, giving $6 \times 27$, which leads to a final answer of 162.

**Question 3:** Evaluate: $\frac{3+5 \times 9}{2^3} - 4 \times (-7)$.

**Answer 3:** The items above and below the division bar should be treated like they are in parentheses, and should be executed before the division is done, even if the operations are on a lower level. Note that the order of operations still applies within an expression inside parentheses, so the expression on the top of the division will be calculated as $3 + 5 \times 9 = 3 + 45 = 48$. This leads to the expression: $\frac{48}{8} - 4 \times (-7)$. Executing the division and multiplication next gives $6 - (-28)$. Subtracting a negative number is like adding its opposite (if I take away your debt, your net worth increases), so $6 - (-28) = 6 + 28 = 34$.

**Question 4:** Evaluate: $\frac{9 - 4 \times 6}{-5} - (2 - 4) \times (-3)$
**Answer 4:** This problem can be started much the same way the last problem started. The items above and below the division bar should be treated like they are in parentheses, and the order of operations should be followed within those parentheses. $9 - 4 \times 6 = 9 - 24 = -15$. We can also calculate $(2 - 4) = -2$, since that is also within parentheses. This makes the entire expression $-\frac{15}{5} - (-2) \times (-3)$. Since there are no powers or roots, we must do the division and multiplication next. When dividing a negative number by another negative number, the result is a positive number, as the negative numbers can divided into each other a positive number of times.

It should be noted that multiplication is not always represented by the symbol: $\times$. Often, the placement of two quantities next to each other is enough to show multiplication. So $5(4 + 3)$ is the same as $5 \times 7$, which would be 35. This could be written $5(7)$ or $(5)7$ or $(5)(7)$, but not $57$, since that looks too much like fifty-seven. This is often used with variables, so that “$8x$” is 8 times the unknown quantity $x$.

### Fractions

I know calculations with fractions have scared away many math students. The top of a fraction is called the **numerator** and bottom of a fraction is called the **denominator**. There are three important facts to remember when manipulating fractions.

1) **Multiplying or dividing both the top and bottom of a fraction by any number (other than zero) will not change the value of the fraction.** So $\frac{2}{12} = \frac{3}{4}$, from dividing both the top and bottom by 3. And $\frac{7}{20} = \frac{35}{100}$, from multiplying the top and bottom by 5. When no integer divides evenly into both the numerator and denominator, the fraction is said to be in **lowest terms**.

2) **When multiplying fractions, multiply the numerators together and multiply the denominators together.** $\frac{2}{3} \times \frac{5}{8}$ would be $\frac{2 \times 5}{3 \times 8} = \frac{10}{24}$. This fraction can be reduced to lowest terms by dividing both the top and bottom by two, converting $\frac{10}{24}$ into $\frac{5}{12}$. If you are multiplying an integer by a fraction, write the integer as a fraction with a denominator of one, then multiply the fractions as before. So $4 \times \frac{3}{14} = \frac{4 \times 3}{14} = \frac{12}{14} = \frac{6}{7}$. This fraction can be reduced to lowest terms by dividing the top and bottom of the fraction by two, giving $\frac{6}{7}$.
3) When adding or subtracting fractions, the denominators must match. Just as you cannot add apples to oranges, you cannot add fifths to sevenths. Converting many different denominators to a single number is a process called finding a common denominator. Do this by multiplying the top and bottom of one of the fractions, or possibly both fractions, by a number to make them match; then the fractions can be added or subtracted by adding or subtracting the numerators.

Adding $\frac{1}{5}$ and $\frac{7}{10}$ would require you to multiply the top and bottom of the first fraction by two to make the denominators match, giving $\frac{1}{5} + \frac{7}{10} = \frac{2}{10} + \frac{7}{10} = \frac{9}{10}$. Subtracting $\frac{1}{5}$ from $\frac{7}{10}$ is harder since 5 and 7 do not divide evenly into each other. Both 5 and 7 divide into 35, so we need to convert the fractions into fractions with denominators of 35: $\frac{4}{7} - \frac{1}{5} = \frac{4 \times 5}{35} - \frac{1 \times 7}{35} = \frac{20}{35} - \frac{7}{35} = \frac{13}{35}$. When adding or subtracting integers with fractions, write the integer as a fraction with a denominator of one, and then proceed with converting to a common denominator:

$2 - \frac{4}{7} = \frac{2 \times 7}{7} - \frac{4}{7} = \frac{14}{7} - \frac{4}{7} = \frac{10}{7}$. It is best not to write this as $1 \frac{3}{7}$, since this could be interpreted as “1 times $\frac{3}{7}$.” Writing the answer as $\frac{10}{7}$ avoids this problem.

**Question 6:** Evaluate: $\frac{31}{5} - 4 \times \frac{5}{6}$

**Answer 6:** To multiply 4 and $\frac{5}{6}$, write the integer 4 as the fraction $\frac{4}{1}$, then multiply $\frac{4}{1}$ by $\frac{5}{6}$ to get $\frac{20}{6}$, according to the second fact mentioned above concerning fractions. Using the first fact mentioned above this fraction can be reduced by dividing the top and bottom of the fraction by two to get $\frac{10}{3}$. The expression now stands as $\frac{31}{5} - \frac{10}{3}$. Combining these fractions requires a common denominator, as mentioned in the third fact above. A common denominator of 15 will work, resulting in a final calculation of $\frac{31}{5} - \frac{10}{3} = \frac{93}{15} - \frac{50}{15} = \frac{43}{15}$.

Writing the final answer as $2 \frac{13}{15}$ is misleading since placing a two next to $\frac{13}{15}$ could imply multiplication, meaning $2 \times \frac{13}{15}$. Writing the answer as $\frac{43}{15}$ creates no such ambiguity.

**Exercises**

1. Calculate: $7 + 5 \times 6$
2. Calculate: $15 - 2 \times 7$
3. Calculate: $5 - 4 \times (3 - 9)$
4. Calculate: $\frac{4 - 10}{3} \times (5 - 2^3)$
5. Calculate: $5(3 + 4^2) - \frac{7 - 3}{2}$
6. Calculate: $8\sqrt{25} - 9 + \frac{15}{1+4}$
7. Calculate: \( \left( \frac{7 + 2}{2} \right)^2 - 7 \times \frac{2}{3} \)

8. Calculate: \( \sqrt{4 \times 9} - \frac{4 \times 2}{5} + 1 \)
0.2 Variables

Variables, sometimes called *unknowns*, are quantities that are not yet determined. These quantities are represented by letters until they can be determined (in some cases they can never be determined). If the same letter appears more than once in the same expression, it will represent the same unknown quantity.

The same rules governing the order of operations apply to expressions with variables, but be careful when simplifying these expressions, as you cannot always carry out a calculation involving a variable. One rule you will find useful with simplifying expressions with variables is the *Distributive Law*.

To help illustrate the distributive law, note that multiplication can be “seen” on a two dimensional grid. One can picture $3 \times 6$, by picturing three rows of six objects. Similarly, picturing $5 \times (4 + 3)$, you can see one block of $5 \times 4$ objects and another block of $5 \times 3$ objects. From this we conclude that $5 \times (4 + 3) = 5 \times 4 + 5 \times 3$. This law can be used with numbers or variables or any combination of the two.

The *Distributive Law* states that for any quantities $a$, $b$, and $c$:

$$a \times (b \pm c) = (a \times b) \pm (a \times c)$$

The name comes from the appearance of distributing the multiplied quantity, in this case $a$, across a sum or difference.

**Question 1:** Simplify: $3(y + 2)$

**Answer 1:** Knowing that the number 3 next to the quantity $y + 2$ means multiplication, we can apply the distributive law to get the sum of 3 times $y$ and 3 times 2, which would be $3y + 6$.

**Question 2:** Simplify: $(4 - z)5 + 7$

**Answer 2:** The order in which things are multiplied does not change the distributive law, so $4 - z$ times 5 is no different from 5 times $4 - z$. The result of this would be the difference of 5 times 4 and 5 times $z$, which still must be added to 7. This gives: $20 - 5z + 7$. While the variable $z$ will not allow this quantity to be calculated completely, we can still add 20 and 7 to get a final answer of $27 - 5z$.

**Question 3:** Simplify: $2 - 6(x - 1)$

**Answer 3:** Be careful applying the distributive law here. If you distribute the 6 across $(x - 1)$, you get $(6x - 6)$. However, since this expression is being subtracted from 2, you must subtract both pieces from 2. You do not get $2 - 6x - 6$, but instead you get $2 - (6x - 6)$, which is $2 - 6x + 6$. One way to avoid this error is to consider subtraction as the addition of a negative. $8 - 5$ is the same as $8 + (-5)$. Similarly, we can write $2 - 6(x - 1)$ as $2 + (-6)(x - 1)$. Seeing the problem this way makes it clear that $-6$ is what is distributed, not just 6. From the expression $2 - 6x + 6$, add the two and six, giving a final expression of $8 - 6x$.

**Question 4:** Simplify: $6x + 9x$

**Answer 4:** In this case, we will use the distributive law, but in reverse. Instead of distributing a multiplied quantity across a sum or difference, we can recognize this expression as the result of having already distributed the variable $x$, across the sum $6 + 9$. So $6x + 9x = (6 + 9)x = 15x$. 

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The technique used in this problem is often called factoring. If you take another course in algebra, you will learn many techniques for finding ways to simplify expressions through factoring.

**Question 5:** Simplify: \(12(p - q + 2) - 2(3 - p) + 10(q + 1)\)

**Answer 5:** The first step is to start distributing. Distribute the 12 across \(p - q + 2\) to get \(12p - 12q + 24\). Then distribute the \(-2\) across \(3 - p\) to get \(-6 + 2p\). Finally distribute the 10 across \(q + 1\) to get \(10q + 10\). Putting all this together gives the expression: \(12p - 12q + 24 - 6 + 2p + 10q + 10\). But we are not finished yet, since we can use the distributive law to factor. Rearranging the terms to put the \(p\)'s and \(q\)'s together, we get: \(12p + 2p - 12q + 10q + 24 - 6 + 10\). Collecting the terms with \(p\)'s gives \(12p + 2p = (12 + 2)p = 14p\). Collecting \(q\)'s, we get \(-12q + 10q = (-12 + 10)q = -2q\). And the remaining constants are \(24 - 6 + 10 = 28\), giving a final answer of \(14p - 2q + 28\).

**Question 6:** Simplify: \(5x - 2 + \frac{5 - 3x}{2}\)

We learned in the last section that items over and under a division bar should be treated as parentheses, so we should subtract \(3x\) from \(5\) before dividing by \(2\). The problem is that we cannot simplify \(5 - 3x\) any further, creating an apparent roadblock to any efforts to simplify the expression. However, we can use the distributive law if we look at this fraction in another way. Just as we can view subtraction as adding a negative, making \(8 - 5\) the same as \(8 + (-5)\), we can view division as multiplication by a reciprocal, so \(\frac{13}{4}\) is the same as \(\frac{1}{4} \times 13\). The reciprocal of a number \(a\) is the number \(\frac{1}{a}\), which, when multiplied by \(a\), gives 1.

**Answer 6:** Writing the division by two as multiplication by one half, we get to \(5x - 2 + \frac{1}{2}(5 - 3x)\). Distributing the one half, we get \(5x - 2 + 2\frac{3}{2}x\). Collecting the terms with \(x\)'s and applying the distributive law leads to \((5 - \frac{3}{2})x\) while the remaining constants are \(-2 + \frac{5}{2}\). Using common denominators, this becomes \((\frac{10}{2} - \frac{3}{2})x = \frac{7}{2}x\). As mentioned earlier, it is better to use \(\frac{7}{2}\) than \(\frac{3}{2}\) to avoid confusion over thinking that \(\frac{3}{2}\) means three times one half.

It should be pointed out that this trick will not be effective with complicated denominators. While it is true that \(\frac{6}{4 + 2} = \frac{1}{4 + 2} \times 6\), the multiplication cannot be distributed over the denominator, so this is not equal to \(\frac{6}{4} + \frac{6}{2}\). Similarly, avoid the temptation to simplify \(\frac{9}{4 + x} = \frac{9}{4} + \frac{9}{x}\). The expression \(\frac{9}{4 + x}\) cannot be simplified any further.

**Question 7:** Simplify: \(a + ab\)

**Answer 7:** At first glance, it doesn’t appear the distributive law will be useful. There is nothing to distribute, and the expression does not look large enough to find something to factor. However, it sometimes helps to write a single variable as one times that variable, in particular \(a = 1a\). This allows the \(a\)'s to be collected while leaving the one behind to hold its place. You may also change the order of any multiplication, so \(ab\) is the same as \(ba\). Using these tricks, we get \(a + ab = 1a + ba = (1 + b)a\).
Question 8: Simplify: \((x + 3)(2 - x)\)

Answer 8: The distributive law may still be applied when multiplying by a more complicated expression. Distribute \(x + 3\) across \(2 - x\) and get: \(x(2 - x) + 3(2 - x)\). Then distribute the \(x\) and the 3 across \(2 - x\) to finish the process: \(2x - x^2 + 6 - 3x\). Combining the \(2x\) and \(-3x\) gives \((-x^2 - x + 6)\), which can be written \(-x\), so the final answer would be \(-x^2 + x + 6\).

Another way to track the multiplication of two expressions each having two terms like \((x + 3)(2 - x)\) is illustrated in the box to the right. The product of the first terms is \(2x\), the product of the outside terms is \(-x^2\), the product of the inside terms is 6, and the product of the last terms is \(-3x\). This process is another way to guarantee that you have each of the four possible terms that are created when multiplying two expressions each having two terms.

Exercises

1. Simplify: \(4(b + 3)\)
2. Simplify: \(6(2 - h)\)
3. Simplify: \(3(x + 7) - 2(5 - x)\)
4. Simplify: \(8(3 + y) - 5(y + 5)\)
5. Simplify: \(a(b + 7) - 2b(3 - a)\)
6. Simplify: \(p(4 - q) + q(2 + 4p - 3q) + q^2\)
7. Simplify: \(\frac{h + k + 4}{2} - (h - k + 4)\)
8. Simplify: \(2 + d - \frac{3d + 8}{6}\)
9. Simplify: \((3 + a)(2 - b)\)
10. Simplify: \(4 - x + (x + 2)(2x + 3)\)

“FOIL”

In answer 8, a product like \((a+b)(c+d)\) was simplified using two steps with the distributive law. Another way to picture this multiplication is seen here:

\[
\begin{array}{cc}
\text{a} & \text{b} \\
\text{c} & \text{ac} & \text{bc} \\
\text{d} & \text{ad} & \text{bd}
\end{array}
\]

\((a+b)(c+d) = ac + ad + bc + bd\)

The acronym “FOIL” is used to help create the four terms of this product. The product of the First terms is \(ac\), the product of the Outside terms is \(ad\), the product of the Inside terms is \(bc\), and the product of the Last terms is \(bd\).
1 Problem Solving

1.1 Translating Expressions

When it comes to solving problems, many algebra texts spend pages upon pages describing all the various rules of the algebra game, but they talk very little about how to translate problems into the language of algebra. For this reason, one of the biggest hurdles students face in solving a problem can be how to get started. The secret for how to get started is to examine phrases of your problem and translate them into mathematical expressions.

**Question 1:** Andy, Britney, Conrad, Drew, and Emma all bought big-screen televisions from different companies. Britney paid $50 less than Andy. Conrad paid twice as much as Andy. Drew paid $100 more than Conrad. Emma paid three times as much as Britney. If Andy spent $x$ dollars on his television, find expressions, in terms of $x$, for the amounts that the other buyers paid, and find an expression for the total that all five paid for their televisions.

**Answer 1:**

Britney: $x - 50$, since she paid $50$ less than Andy, who paid $x$.

Conrad: $2x$, since he paid twice as much as Andy, who paid $x$.

Drew: $2x + 100$, since she paid $100$ more than Conrad, who paid $2x$.

Emma: $3(x - 50)$, since she paid three times as much as Britney, who paid $x - 50$.

Using the distributive law, Emma’s total could be written as: $3x - 150$.

Grand total: $(x) + (x - 50) + (2x) + (2x + 100) + (3x - 150)$.

Reordering this sum gives: $x + x + 2x + 2x + 3x - 50 + 100 - 150$.

Using the distributive law to collect the $x$’s and constants, we get: $9x - 100$.

One must pay very close attention to the phrases to be sure to translate them correctly. Key phrases, like “more than” or “less than” indicate addition and subtraction, while phrases like “twice as much” or “times as much” would hint at multiplication. Some other common phrases appear in this chart.

<table>
<thead>
<tr>
<th>Phrase</th>
<th>Translation</th>
</tr>
</thead>
<tbody>
<tr>
<td>“3 more than a number”</td>
<td>$x + 3$</td>
</tr>
<tr>
<td>“The sum of 3 and a number”</td>
<td>$x + 3$</td>
</tr>
<tr>
<td>“3 added to a number”</td>
<td>$x + 3$</td>
</tr>
<tr>
<td>“The total of 3 and a number”</td>
<td>$x + 3$</td>
</tr>
<tr>
<td>“3 less than a number”</td>
<td>$x - 3$</td>
</tr>
<tr>
<td>“A number minus 3”</td>
<td>$x - 3$</td>
</tr>
<tr>
<td>“3 times as much as a number”</td>
<td>$3x$</td>
</tr>
<tr>
<td>“The product of 3 and a number”</td>
<td>$3x$</td>
</tr>
<tr>
<td>“A number multiplied by 3”</td>
<td>$3x$</td>
</tr>
<tr>
<td>“3 of a number”</td>
<td>$3x$</td>
</tr>
<tr>
<td>“One third of a number”</td>
<td>$\frac{x}{3}$ or $\frac{1}{3}x$</td>
</tr>
<tr>
<td>“A number divided by 3”</td>
<td>$\frac{x}{3}$ or $\frac{1}{3}x$</td>
</tr>
<tr>
<td>“The ratio of a number to 3”</td>
<td>$\frac{x}{3}$ or $\frac{1}{3}x$</td>
</tr>
</tbody>
</table>

**Question 2:** Two numbers add to 15. If $x$ is one of the numbers, what is the other number in terms of $x$?

**Answer 2:** In this case, the puzzle is to figure out what can be added to $x$ to get 15, or $x + ? = 15$. This mystery needs to be the difference between 15 and $x$, which is $15 - x$. To check, add this to $x$ and get $x + (15 - x) = 15 - x + x = 15$.

**Question 3:** Let $x$ be the smaller of two consecutive integers, give an expression for the quantity that is four times as big as the larger of the two consecutive integers.

**Answer 3:** If two integers are consecutive, meaning next to each other in size, then the larger one must be exactly one bigger than the smaller one. Thus, if the smaller integer is $x$, the larger integer is $x + 1$. Four times as big as this would be $4(x + 1)$, which simplifies to $4x + 4$ using the distributive law.
In the case of consecutive even or consecutive odd integers, the difference between these would be two, so if the smaller of two consecutive odd integers is \( x \), the larger would be \( x + 2 \).

**Question 4:** Two numbers add to 30. If the smaller number is \( x \), find an expression equal to the product of four less than the smaller and three times the larger.

**Answer 4:** There is quite a bit going on in this problem, so dissect all the pieces carefully. The final expression is “the product of” something, so we will be multiplying objects together. One object is “four less than the smaller,” and since the smaller is \( x \), this is \( x - 4 \). The other object is “three times the larger.” If the smaller number is \( x \), and the numbers add to 30, the larger must be \( 30 - x \). Three times this quantity would be \( 3(30 - x) \), which simplifies to \( 90 - 3x \). The product of these would be \((x - 4)(90 - 3x)\). This can be simplified by applying the distributive law twice. First distribute the \( x - 4 \), giving \( 90(x - 4) - 3x(x - 4) \), then distribute the 90 and \(-3x\) over the \( x - 4 \), giving: \( 90x - 360 - 3x^2 + 12x \). You would arrive at the same four-term expression by using the “FOIL” device from section 0.2. To complete the problem, combine the \( x \) terms, giving a final expression of: \( 102x - 360 - 3x^2 \).

**Question 5:** Let \( x \) be the larger of two consecutive odd integers. Find an expression for the sum of three times the smaller odd integer and two more than half of the larger odd integer.

**Answer 5:** Once again, be careful while you decode the expression. The final expression is “the sum of” something, so we will be adding objects together. One object is “three times the smaller odd integer.” Since the larger odd integer is \( x \), the smaller must be \( x - 2 \), and three times this is \( 3(x - 2) \) or \( 3x - 6 \). The other object is “two more than half of the larger odd integer,” so this is \( 2 + \frac{x}{2} \). Adding these two objects gives \( 3x - 6 + 2 + \frac{x}{2} \). Combining \(-6 + 2 \) amounts to \(-4 \).

Combining \( 3x \) and \( \frac{x}{2} \) requires the division by two to be written as multiplication by a half, which makes the expression look like: \( 3x + \frac{x}{2} = 3x + \frac{1}{2}x = \left(3 + \frac{1}{2}\right)x \). To finish simplifying this expression, use a common denominator: \( 3 + \frac{1}{2} = \frac{6}{2} + \frac{1}{2} = \frac{7}{2} \). The final answer would now be \( \frac{7}{2}x - 4 \).

**Question 6:** Yvonne has three more collectable stuffed animals than Zachary. Yvonne’s animals are worth \$2 each and Zachary’s are worth \$3 each. If Zachary owns \( z \) stuffed animals, how much is the combined collection worth?

**Answer 6:** The first piece of information indicates a sum. Yvonne has “three more” than Zachary, who has \( z \) animals, so Yvonne has \( z + 3 \) animals. The second piece of information is a well-disguised multiplication. If Zachary’s animals are worth \$3 each, and there are \( z \) of them, they are worth \( 3z \) dollars (notice the “of” as a phrase indicating multiplication). Similarly, since each of Yvonne’s \( z + 3 \) animals is worth \$2, her total worth is \( 2(z + 3) \) or \( 2z + 6 \) dollars. This gives a total of \( 3z + (2z + 6) \), or \( 5z + 6 \) dollars.

Anytime a problem involves a quantity of objects and a price per object, the total value will be calculated as a product.

<table>
<thead>
<tr>
<th>Value of a collection</th>
<th>Quantity of objects</th>
<th>Price per object</th>
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</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
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</tbody>
</table>

**Exercises**

1. Francine paid 15 dollars less than George for her airline ticket. If George paid \( x \) dollars for his ticket, give an expression for the amount Francine paid in terms of \( x \).
2. Isaac paid four times as much as Haley for his autographed copy of a Broadway program. If Haley paid \( x \) dollars for her program, give an expression for the amount Isaac paid in terms of \( x \).

3. Judy was quoted a price from Keith’s Konfections for a wedding cake over the phone. But when she went to Larry’s Larder, she was offered a better price, which was only 20 dollars more than half the price she was quoted from Keith’s Konfections. If the quoted price from Keith’s Konfections was \( q \) dollars, give an expression for the cost from Larry’s Larder in terms of \( q \).

4. Mike is looking to buy a used computer. Olivia will sell hers for 100 dollars less than twice the price Nancy is offering. If Nancy is offering a price of \( n \) dollars, give an expression for Olivia’s price in terms of \( n \).

5. Together, Peter and Quincy own 9 collectable baseball cards. Peter’s cards are worth $18 each and Quincy’s cards are worth $20 each. If Peter owns \( p \) baseball cards, give an expression in terms of \( p \) for the value of their combined collection.

6. Rachel and Sasha bought 7 sweaters. Rachel bought hers for $25 each, while Sasha paid only $15 for each sweater. If Rachel bought \( r \) sweaters, give an expression in terms of \( r \) for the total amount spent by both of them.

7. Let \( x \) be the smaller of two consecutive integers. Give an expression for the product of the two integers.

8. Let \( y \) be the larger of two consecutive even integers. Give an expression for the sum of two times the smaller and three times the larger.

9. Let \( z \) be the largest of three consecutive odd integers. Give an expression for twice the sum of all three integers.

10. Let \( w \) be the smallest of four consecutive integers. Give an expression for the product of six times the largest integer and three more than the next largest integer.

Is It Reasonable?

11. Knowing the sum of three numbers, if you discover the value of the largest and smallest number, is it reasonable to expect that you can figure out the exact value of the middle number?

12. Knowing the sum of four numbers, if you discover the value of the largest and smallest number, is it reasonable to expect that you can figure out the exact values of the middle two numbers.

Advanced Applications:

13. Let \( h \) be the smaller of two integers that multiply to 24. Give an expression for three times a number one higher than the larger integer.

14. Let \( i \) be the largest of nine consecutive even integers. Give an expression for the product of the middle integer and the smallest integer.
1.2 Equations

Equations simply show the equivalence of two expressions. If Victor has $v$ cats, and Wilma has $w$ cats, then the equation $v = w + 3$ indicates that the total number of Victor’s cats is three more than Wilma’s total. Notice that what distinguishes equations from expressions is the verb: *is*. When building an equation from a problem, some form of the verb *is* will usually indicate where two expressions are equal.

Some equations, like $v = w + 3$, cannot be solved alone. There is not enough information to figure out what values are represented by the variables. Other equations, like $2x + 4 = 10$, can be solved. A little investigation leads to the fact that if $x$ is equal to 3, then the equation will be true, since $2 \times 3 + 4 = 10$, but if $x$ is equal to any other number, then the equation will not be true. In this section we will learn techniques for solving basic equations. Should you find yourself in a more advanced mathematics course, you are likely to learn to solve more complicated equations.

In addition to the techniques you learned to manipulate expressions with the distributive law, you can manipulate an equation by performing an operation on both sides of an equation. For example, if we know that $v = w + 3$, we can add five to both sides and get a new equation $v + 5 = w + 8$ that still must be true. The trick to solving an equation is picking an operation to perform that is useful in solving the puzzle.

**Question 1:** Solve: $2x + 4 = 10$.

I know we just solved this one a couple paragraphs ago, but I want to use this simple equation to display the technique. What makes this equation simpler than most is that there is only one variable appearing in one place in the equation. If we can find operations to perform on this equation that will isolate the variable, leaving an equation that looks like “$x$ is something,” where “something” contains no other $x$‘s, we will have solved the equation. When choosing operations, try to “undo” the operations acting on $x$.

**Answer 1:** To isolate the $x$, first undo the addition of four by subtracting four from both sides. Subtracting four from $2x + 4$ leaves $2x$, and subtracting four from ten leaves six, making the equation $2x = 6$. Next, we can undo the multiplication by two by dividing both sides by two. Dividing $2x$ by two leaves just $x$, and dividing six by two gives 3. Having isolated the variable, we can conclude the solution is $x = 3$.

A nice feature of equations is that answers are easy to verify. By plugging in your suspected answer back in the original equation in place of the variable, $2 \times 3 + 4 = 10$, we can verify quickly that the answer is correct.

**Question 2:** Solve: $5 - 6(x - 2) = 23$

**Answer 2:** Once again, we have only one appearance of a single variable, but the expression containing the variable can be simplified. By distributing the $-6$ across $x - 2$, we get $-6x + 12$, so the left side of this equation becomes $5 - 6x + 12$, or $17 - 6x$. The entire equation now appears as $17 - 6x = 23$. Now we can start undoing the operations around the variable. We can remove the 17 by subtracting 17 from both sides. Next, we can undo the multiplication of $-6$ by dividing by $-6$. Having isolated the variable, we can conclude the solution is $x = -1$. To check the answer, insert $x = -1$ back into the original equation. The left side of the equation becomes $5 - 6(-1 - 2) = 5 - 6(-3) = 5 - (-18) = 5 + 18 = 23$, which agrees with the right side.
Question 3: Solve: $6x - 13 = 4(x - 2)$

In this equation, the variable appears in two places. This means that simply adding 13 to both sides to get $6x = 4(x - 2) + 13$ and then dividing by six to isolate the $x$ will not solve the problem. Having $x$ equal to another expression involving the same variable $x$ does not yield an answer. We must first find a way to combine the terms containing $x$’s before applying the principles of isolation.

**Answer 3:** Start by applying the distributive law to simplify the expressions: $4(x - 2) = 4x - 8$, so the equation becomes $6x - 13 = 4x - 8$. To combine the $x$’s, we first need to clear one side of all terms containing $x$. We can do this by subtracting $4x$ from both sides. Now we can rearrange $6x - 13 - 4x$ as $6x - 4x - 13$, and then use the distributive law to combine the terms with $x$’s: $6x - 4x - 13 = (6 - 4)x - 13 = 2x - 13$. Now we can continue as before, isolating the $x$. Add 13 to both sides to get $2x = 5$, then divide both sides by 2 to get $x = \frac{5}{2}$.

This can be checked by calculating each side of the equation with $x$ replaced by $\frac{5}{2}$. Left side: $6x - 13 = 6(\frac{5}{2}) - 13 = \frac{30}{2} - 13 = 15 - 13 = 2$.

Right side: $4(x - 2) = 4(\frac{5}{2} - 2) = 4\left(\frac{3}{2} - \frac{4}{2}\right) = 4\left(\frac{1}{2}\right) = \frac{4}{2} = 2$. So the equation checks.

**Question 4:** Solve: $3(x - 2) = 2(x + 1) + x$

It should be pointed out that not every equation must have a solution.

**Attempted Answer 4:** Using the distributive law to simplify where possible, we get $3x - 6 = 2x + 2 + x$. Collecting $x$’s on the right side brings this to $3x - 6 = 3x + 2$. Finally, subtracting $3x$ from both sides yields $-6 = 2$, which will never be true, regardless of what value $x$ takes. This equation has no solution.

**Question 5:** Solve $4x + y = 3 - 2z$ for the variable $x$.

In some cases, you may find the need to solve an equation with multiple variables. While you will not get a numeric answer as you did in most of the other questions in this section, you will get an expression using the other variables in your answer. Follow the same steps you would use to isolate the variable as before.

**Answer 5:** Undo the addition of $y$ by subtracting $y$ from both sides, then undo the multiplication by $4$ by dividing both sides by $4$. We end up with $\frac{3-2z-y}{4}$, which is quite a mess, but this is the best answer we can find.

**Question 6:** Solve $5a - 3b + 2 = 2a + 6b - 4$ for the variable $a$.

**Answer 6:** Since the variable, $a$, appears in two places, we first must remove the terms with $a$’s from one of the sides of the equation. Subtracting $2a$ from both sides will remove all the terms with $a$’s from the right side. Combining terms with $a$’s on the left side leaves this side as $3a - 3b + 2$. The $a$ can now be isolated by adding $3b$ to both sides and subtracting two from both sides, leaving an equation of $3a = 9b - 6$. Dividing both sides by three would leave $a = \frac{9b - 6}{3}$, but if you imagine division by three as multiplication by $\frac{1}{3}$, you can use the distributive law to simplify $\frac{1}{3}(9b - 6)$ to $3b - 2$. 

<table>
<thead>
<tr>
<th>Expression</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$5a - 3b + 2 = 2a + 6b - 4$</td>
<td>$a = \frac{9b - 6}{3}$</td>
</tr>
</tbody>
</table>
Exercises

1. Solve: $7x - 11 = 3$
2. Solve: $4x + 5 = 17$
3. Solve: $5(x - 2) = 2x + 5$
4. Solve: $8x + 11 = 3(x + 2)$
5. Solve: $3x - 2 = 3(x + 4)$
6. Solve: $2(2 + 3x) = 6x - 2$
7. Solve for $p$: $4p + q = 9q - 12$
8. Solve for $b$: $3b - 2c = 5b + 7c - 5$

Is It Reasonable?

9. Is it reasonable to assume that every equation that only uses a single variable (possibly in multiple places) will always have a solution?
10. Is it reasonable to assume that doing the same operation (correctly) to both sides of a true equation will always result in a true equation?

Advanced Applications:

11. For what value of $k$ will the following equation have no solution for $x$? $3(4 + 2x) = k(x + 5)$
12. Solve for $j$: $(j + i)(j - 2) = (3 - j)(2 - i)$
1.3 Translating Equations

This is where many applications arrive. Some of these problems may seem trivial, but you can apply these ideas to solve more complicated problems when they arise. Each of these problems consists of three steps:

1] Translate the problem into an equation.  2] Solve the equation.  3] Translate the answer back into the language of the original problem.

**Question 1:** The sum of two consecutive integers is 13. What are the integers?

Your first task in translating the problem is to decide which quantity to use as your variable. Since you are considering two consecutive integers, you could assign the smaller integer $x$ and the larger $x + 1$, or you could assign the larger integer $x$, which would make the smaller integer $x - 1$. Either choice will lead to the correct answer, so pick one.

**Answer 1:** Let $x$ be the smaller integer, making the larger integer $x + 1$. To create the equation, read carefully. The sum of the two integers is 13. The sum of the two integers would be $x + x + 1$, or $2x + 1$. Since this is 13, the equation becomes $2x + 1 = 13$. To solve this equation, isolate the variable by subtracting one from both sides, then dividing by two. Finally, to translate the answer back into the language of the original problem, notice that the problem wants the values of both integers. Since the integers are $x$ and $x + 1$, and the equation was solved by $x = 6$, the integers must be 6 and 7.

Had we chosen to make the integers $x$ and $x-1$, our equation would have solved to $x = 7$ (try it and see), but this would still give the same final answer, namely that the integers must be 6 and 7.

**Question 5:** Two numbers add to 15. One number is four times as big as the other. Find the smaller number.

**Answer 5:** Since we have two numbers adding to 15, if one of them is $x$, the other must be $15 - x$. Reading the next sentence gives the equation. “One number is four times as big as the other.” It shouldn’t matter which number you use in which place, the final answer will not change. Choosing $x$ as “one number” and $15 - x$ as “the other” gives an equation of: $x = 4(15 - x)$. Simplifying the right side with the distributive law makes this: $x = 60 - 4x$. To solve this, remove the $x$ terms from one side of the equation, combine them on the other side, and then finish solving to get $x = 12$. To translate the problem back, notice it is asking for the smaller number. If $x = 12$, then the two numbers, being $x$ and $15 - x$, would be 12 and $15 - 12 = 3$. Since three is the smaller number, that is the answer.

**Question 6:** Carla went shopping for bananas and apples. Each banana was 35 cents and each apple cost 50 cents. Carla bought a total of eight pieces of fruit, which cost a total of $3.55. How many bananas did she buy?

Before you panic over a problem like this, look at the phrases to translate each piece of the puzzle, then look hard for the equation. Since the problem is asking for the number of bananas, this would seem a logical choice for the variable. If you ever get tired of using $x$, feel free to try out other letters.

**Answer 6:** Let $b$ be the number of bananas Carla bought. Since she bought a total of eight pieces of fruit, the number of apples must be $8 - b$. Now set up the equation. The word is does not appear, but from the last sentence before the question, we are told that $3.55$ is what she spent on fruit. All we need is an
expression for how much she spent, in terms of \( b \), which we can then set equal to $3.55. Each banana costs 35 cents, or $0.35, making the cost of the bananas \( 0.35b \) (price per banana times number of bananas). Each apple costs 50 cents, or $0.50, making the cost of the apples \( 0.50(8 - b) \) (price per apple times number of apples). Adding these quantities and setting the total to 3.55 creates the equation: \( 0.35b + 0.50(8 - b) = 3.55 \). Applying the distributive law to the left side gives \( 0.35b + 4 - 0.50b = 3.55 \). Combining the terms containing \( b \)'s, we get: \( 4 - 0.15b = 3.55 \). We are ready to solve by subtracting four from both sides and then dividing both sides by \(-0.15\), giving a solution of \( b = 3 \). Since the problem asks for the number of bananas, which we conveniently chose for the variable, the answer we seek is simply three bananas.

**Question 7:** A collector has coins in three denominations: some are worth $1, some are worth $2, and some are worth $5. The total value of the collection is $100. The collector has five more $2 coins than $5 coins, and the number of $1 coins is exactly half the number of $5 coins. How many of each coin does the collector have?

We have several choices for a variable. We could use the number of $1 coins, the number of $2 coins, or the number of $5 coins. Any choice should lead to a correct answer.

**Answer 7:** Let \( x \) be the number of $5 coins. Since the number of $2 coins is five more than this, there are \( x + 5 \) of the $2 coins. Since the number of $1 coins is half of the number of $5 coins, this is \( \frac{1}{2} \) or 0.5\( x \). Since the total value of a collection is equal to the value of one object multiplied by the number of objects, the total value of the $5 coins would be five times \( x \), or \( 5x \); the total value of the $2 coins would be two times \( (x + 5) \), or \( 2(x + 5) \); and the total value of the $1 coins would be one times \( 0.5x \), which is just \( 0.5x \). Since all three of the groups of coins combined are worth $100, we get a final equation: \( 5x + 2(x + 5) + 0.5x = 100 \). Applying the distributive law to turn \( 2(x + 5) \) into \( 2x + 10 \) and collecting the \( x \)’s yields a final equation of: \( 5x + 2 + 0.5x + 10 = 100 \) or \( 7.5x + 10 = 100 \). Solving, we get \( x = 12 \). To translate back, we have \( x = 12 \) of the $5 coins, \( x + 5 = 17 \) of the $2 coins, and \( 0.5x = 6 \) of the $1 coins.

**Exercises**

1. Two numbers add to 14. One number is two less than three times the other. What are the numbers?
2. The sum of two numbers is 19. One number is one more than twice the other. What are the numbers?
3. Perry bought three more souvenirs than Quentin on their last vacation. If Perry had bought one more souvenir, his total would have been double that of Quentin. How many souvenirs did Perry buy?
4. Not counting the novels she needs for literature, Rachel needed two more textbooks for her courses than Stacy needed. If you count the seven novels Rachel needed for her literature class, Rachel’s total would be four times as big as Stacy’s total. How many textbooks did Stacy need?
5. The Tennessee Titans scored 37 points in a football game. All scores were either touchdowns, worth seven points each, or field goals, worth three points each. If the Titans scored one more touchdown than field goals, how many of each type of scoring play did the Titans get?
6. In Australian Rules Football, teams can score a goal to get six points or score a behind to get one point. If the Sydney Swans scored five more goals than behinds to get 79 points, how many of each type of scoring play did the Swans get?

7. A cash register contains nickels (worth five cents each), dimes (worth ten cents each), and quarters (worth 25 cents each). There are four more nickels than quarters, and twice as many dimes as nickels. If the total value of the coins in the register is $4, how many quarters are in the register?

8. Tara went to the post office and bought $8.15 worth of stamps. She only purchased 12-cent, 15-cent, and 20-cent stamps. She bought five times as many 15-cent stamps as 12-cent stamps, and she bought six fewer 20-cent stamps as 15-cent stamps. How many 12-cent stamps did she purchase?

Is It Reasonable?

9. Ukiah takes two pills every day to control his arthritis pain. Going on a three-week vacation, he packs 30 pills, assuming this will be enough. Is this reasonable?

10. Valerie took 15 kindergarteners to a museum and paid for their lunches. The expense report showed that she spent $600 on the lunches. Is this reasonable?

11. Wilma is paying her babysitter $8 per hour to look after her children. While out on her three-hour shopping trip, where she expects to spend about $150, she withdraws $200 from an Automated Teller Machine to pay for the shopping expenses and the babysitter. Is this reasonable?

12. Xavier can read one page of his economics text book in about two minutes. He budgets an hour of time to read a 25 page chapter. Is this reasonable?

Advanced Applications:

13. Yorick is carrying only $20 bills and $1 bills. If he is carrying more $20 bills than $1 bills, and all the bills together are worth $292, how many of each bill must he have?

14. Zuni rolled several dice, and all of her rolls were either threes or fives. If the sum of the die rolls was exactly 22, how many of each number were rolled?
1.4 Functions

A relation is a list of ordered pairs of objects. Pairs are considered ordered when we can distinguish which object is first and which object is second. In mathematics courses, these objects are usually numbers, but this is not a requirement.

Imagine a room of five people: Ally, Bob, Cory, Dave, and Emily. Each person has a favorite color, and this information could be represented by a relation. The ordered pairs of objects might be:

\{ (Ally, Red), (Bob, Blue), (Cory, Green), (Dave, Red), (Emily, Yellow) \}

From this list we see, for example, that Bob’s favorite color is Blue.

For a list of ordered pairs to be a function, no two pairs can share the same first object. The second object may be repeated, however. This example is a function since no person is allowed to have more than one favorite color, but it is allowed for more than one person to share a favorite color. In this case, both Ally and Dave favor the color red.

Generally, it is easier to read the information in a function through a graph like figure 1.4.1, rather than using a list of pairs. It is standard practice to put all the choices for the first object along the horizontal axis and all the choices for the second object on the vertical axis. Since no person can have more than one favorite color, it is impossible to have one point directly above or below another. Another way of saying this is that no vertical line may go through more than one point. But since two people can share a favorite color, it is allowable for a horizontal line to go through more than one point. Notice that a horizontal line running across “Red” would intersect two points, one at Ally and one at Dave.

Consider another function: Generate a list of all pairs of numbers that add up to five. At first, you might think of the obvious:

\{ (0, 5), (1, 4), (2, 3), (3, 2), (4, 1), (5, 0) \}

But this list is far from complete. We could consider (6, −1), or (−15, 20). But why only work with integers? What about (2.5, 2.5) or (1.256, 3.744)? It will not take long to realize that, unlike the favorite colors above, there are an infinite number of pairs making up this function. As with the favorite color function, this function can be pictured with a graph like figure 1.4.2. Just plotting the six “obvious” pairs and the four other pairs listed above—you may have to look carefully to find (−15, 20)—builds a strong case for believing the infinitely many pairs of numbers that add to five all lie in the same straight line. The arrowheads on the ends of the line demonstrate that the line continues in its direction toward infinity.

Any place a strong pattern is found, there is bound to be a formula that generates the pattern. In this example, what is the condition that must be met for a generic point \((x, y)\) to be part of the function? The answer is that the two numbers must add to five, or \(x + y = 5\). Another way to look at a function is to ask: given the first number, \(x\), how do I calculate the second, \(y\)? Solving \(x + y = 5\) for \(y\) gives \(y = 5 - x\). This formula exactly describes how to calculate the second number, \(y\), when given the first, \(x\). To determine
what number goes with 3, simply plug \( x = 3 \) into the formula \( y = 5 - x \), giving \( y = 5 - 3 = 2 \). To determine what number goes with 100, simply plug \( x = 100 \) into the formula, giving \( y = 5 - 100 = -95 \).

When working with functions, it is common to use \( x \) to label the first object in the pair and \( y \) to label the second object. For this reason, the horizontal axis of a graph is called the x-axis and the vertical axis is called the y-axis.

Graphing

**Question 1:** Construct a graph for the following list of points:
\[
\{(2, 3), (3, 0), (1, 1), (4, 1)\}
\]

**Answer 1:** For each pair, locate where on the x-axis you will find the first number, and where on the y-axis you will find the second number. Where these locations meet will determine where each point is plotted. In this example, since there are only four points, it would not be appropriate to connect the dots with lines, as this would imply there were additional points between the four points you plotted. Note that since the first element of the pairs is never duplicated, this list of points represents a function.

**Question 2:** From the graph in figure 1.4.4, determine the average yearly snowfall and average number of yearly thunderstorms for Atlanta and Vancouver.

**Answer 2:** According to the graph, the point labeled “Atlanta” is located at about (48, 2) and the point labeled “Vancouver” is located at about (6, 24). Since the x-axis is labeled “average annual thunderstorms”, and the y-axis is labeled “average annual snowfall”, we can conclude that the average number of thunderstorms yearly for Atlanta is about 48 and the average annual snowfall is about two inches, while the average number of thunderstorms yearly in Vancouver is close to six and the average annual snowfall is about 24 inches.

**Question 3:** Create a graph for the function \( y = x^2 - x \).

**Answer 3:** A good way to get an idea of what a graph may look like is to determine a few of the pairs that make up the function. Once you create a list, plot these points on a system of axes. In areas where you might not be sure where the graph lies, calculate additional pairs. From calculating the points at integer values of \( x \) from \(-2\) to \(3\), we cannot be sure whether the graph dips below zero or not. By calculating the pair where \( x = \frac{1}{2} \), we see that the graph does dip below zero, since \( y = \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right) = -\frac{1}{4} \). Notice that there exist points of the graph between any sample points we select, so it is appropriate to connect the dots in this example, as seen in figure 1.4.5.

When it comes time to connect the dots, simply connecting them like
a puzzle in a children’s activity book might lead to an inaccurate graph. If you have any uncertainty as to what the graph may look like in a particular area, finding additional pairs can help significantly, as evidenced in the following example:

**Question 4:** Create a graph for the function \( y = \frac{1}{2x-3} \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>-1/6</td>
</tr>
<tr>
<td>0</td>
<td>-1/3</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>1/3</td>
</tr>
<tr>
<td>4</td>
<td>1/5</td>
</tr>
<tr>
<td>1.5</td>
<td>?</td>
</tr>
<tr>
<td>1.4</td>
<td>-5</td>
</tr>
<tr>
<td>1.6</td>
<td>5</td>
</tr>
</tbody>
</table>

**Answer 4:** Using the same approach as the last question, we create a list of some of the pairs that make up the function. In this case, after plotting integer points from -1 to 4, simply connecting the dots does not create a good approximation to the function. Even graphing calculators are prone to making this kind of mistake, so you may want to give your calculator a skeptical eye if something seems not quite right. If you suspect that something different is happening between the \( x \) values of one and two, plot points in between. Trying \( x = 1.5 \) leads to division by zero, and so there is no point to plot. However, plotting points for \( x = 1.4 \) and \( x = 1.6 \), where \( y \) takes the values of -5 and 5, respectively, gives a good indication as to what is happening. This phenomenon is called an asymptote, which you will learn more about if you study algebra in another course.

**Function Notation**

While functions may be written in terms of \( x \) and \( y \), most textbooks and applications use function notation, where the transformation that turns \( x \) into \( y \) is given a name, often another letter, such as \( f \).

The line we examined at the beginning of this section, \( y = 5 - x \), could be written as \( f(x) = 5 - x \). It is important to note that the \( x \) in parentheses in this context does not indicate multiplication; it indicates the parameter of the function \( f \). One could interpret \( f(x) = 5 - x \), as “\( f \) is the function that transforms the number \( x \) into the number \( 5 - x \).” This leads to \( f(4) = 1 \), since \( 5 - 4 = 1 \), and similarly, \( f(6.2) = -1.2 \) and \( f(-67) = 72 \). Notice that the name of the parameter does not affect the nature of the function. While \( x \) is typically used, \( f(x) = 5 - x \) is no different from \( f(n) = 5 - n \), or \( f(\text{cantankerous}) = 5 - \text{cantankerous} \), or even \( f(\xi) = 5 - \xi \).

**Question 5:** Let \( g(z) = \frac{7z + z^2}{1 + z} \). Evaluate \( g(2) \).

**Answer 5:** Note that a parameter can appear multiple times in a function. When calculating \( g(2) \) in this example, replace every instance of the parameter \( z \) with a two and then calculate. \( 7z \) becomes 14, \( z^2 \) becomes four, and \( 1 + z \) becomes three. The quantity 14 plus four divided by three is six, so we get \( g(2) = 6 \).

In rare cases, the parameter might not appear in the function, like \( f(x) = 12 \). When calculating \( f(3) \) in this example, ignore the three and return the value 12, so \( f(3) = 12 \). When calculating \( f(4.629) \), ignore the 4.629 and return the value 12, so \( f(4.629) = 12 \). When calculating \( f(\infty) \), ignore the \( \infty \) and return the value 12, so \( f(\infty) = 12 \). These types of functions are often called constants or constant functions.
Exercises

1. Plot the following points, representing students and hometowns, on a graph:
   \{(Gavin, Erie), (Helen, Scranton), (Ian, Pittsburgh), (Jacquie, Allentown), (Kyle, Philadelphia)\}

2. Plot the following points, representing animals and their classification, on a graph:
   \{(Alligator, Reptile), (Bear, Mammal), (Canary, Bird), (Deer, Mammal), (Egret, Bird)\}

3. Which of the following lists represents a function?
   a. \{(3, 5), (7, 2), (4, 9), (8, 2), (2, 8)\}
   b. \{(4, 4), (5, 6), (7, 9), (4, 2), (2, 1)\}
   c. \{(Fred, Green), (Gina, Blue), (Bernie, Orange), (Olive, Purple)\}

4. Which of the following lists represents a function?
   a. \{(7, 8), (6, 8), (5, 7), (4, 8), (6, 7)\}
   b. \{(1, 3), (3, 3), (7, 3), (0, 3), (12, 3)\}
   c. \{(New York, Yankees), (Chicago, Cubs), (Atlanta, Braves), (New York, Mets)\}

5. Suppose \(f(x) = 3x - 2\).
   a. Calculate \(f(2)\) and \(f(-1)\).
   b. Sketch a graph of the function \(f\).

6. Suppose \(g(x) = 4x - x^2\).
   a. Calculate \(g(5)\) and \(g(-3)\).
   b. Sketch a graph of the function \(g\).

7. Suppose \(h(t) = \frac{1}{1-t}\).
   a. Calculate \(h(4)\) and \(h(1/2)\)
   b. Sketch a graph of the function \(h\).

8. Suppose \(V(p) = \frac{p + 10}{1 - p^2}\)
   a. Calculate \(V(0)\) and \(V(2)\)
   b. Sketch a graph of the function \(V\).

Use figure 1.4.7 for exercises 9 through 14.

9. What is the approximate area of Colorado?

10. What is the approximate area of New Hampshire?

11. What is the approximate elevation of the highest point in Pennsylvania?

12. What is the approximate elevation of the highest point in Washington?

13. Which state has a larger area, South Dakota or Washington?

14. Which state’s highest point is higher, Pennsylvania or Michigan?
Is It Reasonable?

15. A picture of a function will always be a smooth connected line or curve. Is this true?

16. A picture of a function will never intersect a vertical line more than once. Is this true?

Advanced Applications

17. For the function \( f(x) = 4x - 7 \), calculate and simplify \( f(h+3) \).

18. For the function \( D(i) = i^2 - 5i + 2 \), calculate and simplify \( D(2x+1) \).
1.5 Linear Functions

Linear functions are functions that generate a graph of a straight line, making them easy to identify, manipulate and calculate. For these reasons, they are often the functions of choice when modeling real-world data. A linear function can be easily constructed from its slope and y-intercept.

y-intercept

The y-intercept is defined as the value the function has when the independent variable is zero, which is exactly where the graph crosses the vertical axis, often called the y-axis. Looking at the graphs here, the graph in figure 1.5.1 crosses the vertical axis at zero, while the graph in figure 1.5.2 crosses the vertical axis at $\frac{5}{2}$. This applies to finding y-intercepts of non-linear functions as well.

**Question 1:** What is the y-intercept for the line $f(x) = 5x + 2$?

**Answer 1:** Even though this linear function is written without using the variable y, the function notation used indicates that $x$ is the independent variable and the value of $5x + 2$ is the dependent variable. We can write this function as $y = 5x + 2$ if we like. To find the y-intercept, simply set $x = 0$. This gives $y = 5 \times 0 + 2 = 2$, so 2 is the y-intercept.

**Question 2:** What is the y-intercept for the line $5x - 3y = 9$?

**Answer 2:** The relationship between $x$ and $y$ is a little more complicated in this example. We will learn how to verify that the graph for this equation is a line later in this section. To find the y-intercept, the task remains to set $x = 0$ and then solve for $y$. Setting $x = 0$ leaves $-3y = 9$, so $y = \frac{9}{-3} = -3$ is the y-intercept.

Slope

The slope of a linear function is a measure of how steep the graph of the line is. A slope is considered positive if the function rises as the values of the independent variable (usually $x$) increase, while the slope is considered negative if the function falls instead of rises. The faster the function rises or falls, the larger the magnitude of the slope.

The magnitude of the slope is measured by how many units the function rises or falls per unit of change in the independent variable. In the case of a straight line, this number is the same no matter where on the line you measure the slope. For the function in figure 1.5.1, the slope is two, since the function rises two units for every unit change in the independent variable. For the function in figure 1.5.2, the slope is negative one half. Notice that the function falls one unit for every two units of change in the independent variable, meaning the function will fall one-half units for every one unit of change in the independent variable.
**Question 3:** What is the slope of the line containing the points (2, 1) and (5, 3)?

**Answer 3:** While graphing is not a necessity, a picture can be a good way to verify what is happening. Between the two points given, there are three units along the horizontal axis. In that span, the function is rising two units. Examining the ratio of rising two units for every three units along the horizontal axis, we see that the graph must be rising $\frac{2}{3}$ of unit for each unit along the horizontal axis. Thus the slope must be $\frac{2}{3}$.

Note that the change along the horizontal axis can be calculated from the difference in the first coordinates in the pair of points, in this case $5 - 2 = 3$, and the change along the vertical axis can be calculated from the difference in the second coordinates in the pair of points, in this case $3 - 1 = 2$. The ratio of these differences—change in vertical distance divided by change in horizontal distance—will always be equal to the magnitude of the slope.

**Question 4:** What is the slope of the line containing the points (1, 5) and (3, 2)?

**Answer 4:** If we simply construct a ratio between the change in the first coordinate, $3 - 1 = 2$, and the second coordinate, $5 - 2 = 3$, we get a magnitude of $\frac{3}{2}$. Looking at the graph, we see that the line is falling, so the slope must be negative. Our slope is $-\frac{3}{2}$.

To be sure our ratio includes the correct sign, either positive or negative, we should subtract the values in the same order when calculating the differences. If we calculate the difference in the first coordinate by subtracting 1 from 3, note that we are subtracting the value in the first pair from the value in the second pair. To calculate the difference in the second coordinate using the same order, we subtract 5 from 2, giving $2 - 5 = -3$. This leads to a slope calculation of $-\frac{3}{2}$. Note that changing the order on both calculations, $1 - 3 = -2$ and $5 - 2 = 3$, yields the same ratio of $-\frac{3}{2}$. In other words, the order in which you calculate the differences does not matter, as long as the same order is used to calculate both differences. Using this principle, we can create a formula that does not rely on a graph.

The slope, $m$, of a linear function passing through the points $(x_1, y_1)$ and $(x_2, y_2)$ is:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

**Question 5:** What is the slope of the line containing the points (–2, 11) and (5, 8)?

**Answer 5:** Remembering that it does not matter in which order the subtraction is done, so long as both subtractions are done in the same order, we can subtract the values in the first pair from those in the second pair. The differences are $5 - (-2) = 7$ and $8 - 11 = -3$, yielding a slope of $-\frac{3}{7}$.

**Standard Form**

**Question 5:** What is the slope of the line $y = 2x + 3$?

**Answer 5:** We could arbitrarily pick two values of $x$, the easiest being zero and one, find the corresponding values of $y$, namely three and five, and use the slope formula above: $\frac{5 - 3}{1 - 0} = 2$. However, looking at an equation like $y = 2x + b$ for any value of $b$, we see that for each unit increase in $x$, $y$ must increase by 2 units, because the coefficient of $x$ is 2. This leads to the following:
A linear function is in **Standard Form** when it is written as

\[ f(x) = mx + b \text{ or } y = mx + b \]

where \( m \) is the slope of the line and \( b \) is the \( y \)-intercept of the line.

All functions that are lines can be written in standard form. So when you are presented with a relationship between two variables, if you can solve the relationship for a dependent variable and manipulate the equation into standard form, the relationship represents a line. If this cannot be done, the relationship will not be a linear function.

**Question 6:** What is the slope and \( y \)-intercept of the line \( y = 4x - 7 \)?

**Answer 6:** Since this line is in standard form, we can read the slope as four and the \( y \)-intercept as \(-7\). Note that subtracting seven is the same as adding \(-7\), so \(-7\) would correspond to the \( b \) in the standard form equation.

**Question 7:** What is the slope and \( y \)-intercept of the line \( y = 1 \)?

**Answer 7:** This may not look like standard form, but \( y = 1 \) is the same as \( y = 0x + 1 \). The slope is zero and the \( y \)-intercept is one. It is a little strange to have a function that does not depend on its independent variable. The graph of this function would be a horizontal line one unit above the \( x \)-axis.

**Question 8:** What is the slope and \( y \)-intercept of the line \( 5x + 3y = 12 \)?

**Answer 8:** This line is not yet in standard form, so guessing a slope of five and a \( y \)-intercept of 12 would be incorrect. Solving this relationship for the dependent variable \( y \) should reveal what we need. Notice in the last step, where we rearranged the equation to put the term with \( x \) first, and we factored \( x \) out to make the coefficient more obvious. Once this is done, we can determine a slope of \(-\frac{5}{3}\) and a \( y \)-intercept of four.

**Question 9:** What is the slope and \( y \)-intercept of the line represented by \( \frac{x + 2y}{3} = 4 - 3x \)?

**Answer 9:** Since this is not in standard form, we need to apply some algebra to solve for the dependent variable \( y \). To isolate the \( y \), first multiply both sides by 3, then subtract \( x \) from both sides, and finally divide both sides by 2. Distributing the one half across both the 12 and \(-10x\), gives a final equation of \( y = -5x + 6 \). The slope is \(-5\) and the \( y \)-intercept is six.

**Question 10:** What is the slope and \( y \)-intercept of the line \( 5xy = 15 \)?
Answer 10: Once again, we try a little algebra. To isolate the $y$, divide both sides by $5x$. This yields an equation that looks like $y = \frac{15}{5x}$, simplifying to $y = \frac{3}{x}$. The $x$ in the denominator does not allow this equation to fit the standard form for a line equation, and therefore this equation does not represent a line.

Exercises

1. Find the slope of the line containing the points (0, 0) and (3, 6).
2. Find the slope of the line containing the points (0, 0) and (6, 2).
3. Find the slope of the line containing the points (2, 4) and (6, 3).
4. Find the slope of the line containing the points (1, 7) and (3, 3).
5. Find the slope of the line containing the points (–2, 5) and (3, 7).
6. Find the slope of the line containing the points (–1, 6) and (2, –2).
7. Determine the slope and $y$-intercept for the linear function in figure 1.5.7.
8. Determine the slope and $y$-intercept for the linear function in figure 1.5.8.
9. Determine the slope and $y$-intercept for the linear function $y = 4x + 9$.
10. Determine the slope and $y$-intercept for the linear function $y = –2x + 5$.
11. Determine the slope and $y$-intercept for the linear function $f(x) = \frac{7}{2} – 7$.
12. Determine the slope and $y$-intercept for the linear function $f(x) = 2 – \frac{7}{3}$.

Is It Reasonable?

In exercises 13-18, determine which of following represent linear functions. For those that are linear functions, put the equation in standard form and determine the slope and $y$-intercept.

13. $2x + 3y = 18$
14. $5y – 4 = 2x$
15. $2(x + 4) = 3(y + 6)$
16. $(x + 1)y = 3x$
17. $4xy = 16$
18. $3y(x + 1) = 11 – 9x + 3xy$

Advanced Applications

19. Find the slope of the line containing the points (2, –1) and (2, 4)
20. Put the following line equation in standard form: $(y + 3)(2 – x) = 10 – xy$
1.6 Linear Modeling

Now that we can analyze linear functions for their slopes and y-intercepts, the next task is to be able to build linear functions from data we have gathered. Once the decision is made to use a linear model, we can use our data to determine the slope and y-intercept, and from that we can build a linear function of the form \( f(x) = mx + b \). For a line to be determined, we must either have two points, or one point and a slope.

Point and Slope

One nice feature of point-slope problems is that we do not need to do any work to determine the slope; it is already given to us.

**Question 1:** Find the linear function with slope five containing the point (0, –2).

**Answer 1:** The problem is even easier if the point provided is the y-intercept, as it is in this problem. We can recognize the point as the y-intercept because the value of the independent variable is zero. With a slope of five and a y-intercept of –2, the linear function is \( f(x) = 5x - 2 \).

**Question 2:** Find the linear function with slope –3 containing the point (1, 5).

**Answer 2:** In this case, the point provided is not the y-intercept because the value of the independent variable is one, not zero. We still have the slope of –3 given in the problem, so our answer will look something like \( f(x) = -3x + b \), where \( b \) is the yet-to-be-determined y-intercept. To find the y-intercept, we need to note that the function describes the mathematical procedure to convert the independent variable into the dependent variable, and specifically it describes the procedure to convert the input value \( x = 1 \) into the output value \( f(x) = 5 \). Temporarily plugging in \( x = 1 \) and \( f(x) = 5 \) will allow us to solve for the value of \( b \), our y-intercept. In this case \( b = 8 \). Now we “unplug” the values of \( x \) and \( f(x) \) to arrive at our linear function: \( f(x) = -3x + 8 \).

This technique works not only on lines, but other types of functions as well. Almost anytime we are missing only one piece of a function, and we have a point that belongs to the function, we can temporarily plug the data from the point into our mostly complete function and solve for the missing piece.

**Question 3:** Find the linear function with slope 1.4 containing the point (0.3, 2.6).

**Answer 3:** This is no different from question two, except the numbers are not whole. The standard form with a slope of 1.4 will look like \( f(x) = 1.4x + b \). We temporarily plug in \( x = 0.3 \) and \( f(x) = 2.6 \) to solve for \( b \), which gives \( b = 2.18 \). Lastly, we remove the values we plugged in to get \( f(x) = 1.4x + 2.18 \).

Two Points

For problems where we are given two points, the first step becomes determining the slope from those points based on the formula of the previous section, namely \( m = \frac{y_2 - y_1}{x_2 - x_1} \). Recall that it does not matter which of the two points is labeled \((x_1, y_1)\) or \((x_2, y_2)\), we simply need to be consistent when constructing the ratio for the slope.

**Question 4:** Find the linear function containing the points (0, 3) and (2, 4).
Answer 4: Calling the first point \((x_1, y_1)\) and the second point \((x_2, y_2)\), we get a slope of \(m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - 9}{-1 - 2} = \frac{-6}{-3} = 2\). We also notice that one of the points, namely \((0, 3)\) is the \(y\)-intercept. Using a slope of \(2\) and a \(y\)-intercept of \(3\), the linear function is \(f(x) = 2x + 3\).

Question 5: Find the linear function containing the points \((-2, 9)\) and \((1, 0)\).

Answer 5: Again, we begin with the slope calculation: \(m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0 - 9}{1 - (-2)} = \frac{-9}{3} = -3\). Neither of the points given is a \(y\)-intercept, so we temporarily plug in one of the points in our partial function \(f(x) = -3x + b\). It will not matter which point we select since both points must be part of the function. Both calculations are shown here, but only one is needed. It is best to pick the point that makes the calculation easier. In this case, \((1, 0)\) is easier to work with than \((-2, 9)\). The \(y\)-intercept is \(3\) and the function is \(f(x) = -3x + 3\).

Applications

With the mathematical procedure in place, the only challenge remaining in working with applications is identifying the independent and dependent variables. Once this is done, the steps above will lead to linear model.

Question 6: Francine’s Fashions has purchased a computer for $1000. Francine’s accountant estimates that the computer’s value decreases by about $250 each year (this is called depreciation). Construct a linear function to model the depreciation of the computer. Then use this model to calculate the value of the computer after two-and-a-half years, and determine how long it will take for the computer to be worthless.

Answer 6: The first step is to identify the independent and dependent variables. In this scenario we have two quantities that are related: the value of the computer and how old the computer is. Since the value of the computer depends on how old it is, and not the other way around, the age of the computer will be the independent variable, and the value of the computer, which depends on the age, will be the dependent variable.

The next step is to determine if we have a slope and a point or two points. The problem states that the value of the computer decreases by about $250 each year. This indicates a slope: for each unit (one year) of the independent variable (age of computer), the dependent variable (value of computer) decreases by 250 units (dollars). Since the value is decreasing, our slope is negative, so we use \(-250\). The \(y\)-intercept would correspond to the value of the computer when the computer is zero years old, which would be the original purchase price of $1000. Using our slope of \(-250\) and the \(y\)-intercept of 1000, our function is \(f(x) = -250x + 1000\), where \(x\) is the age of the computer in years.

With this model, answering the last parts of the question should be easy. After two-and-a-half years (when \(x = 2.5\)), the value of the computer would be \(f(2.5) = -250 \times 2.5 + 1000 = -625 + 1000 = 375\) dollars. The computer will be worthless when \(f(x) = 0\), so plugging this in the model gives: \(0 = -250x + 1000\), which leads to \(250x = 1000\) and is solved by \(x = 4\), thus the computer will be worthless in four years.

Question 7: Tommy’s T-Shirts will make custom T-shirts for $7 each, plus a fixed cost (which we do not know yet) for setting up the machinery. After placing an order for 50 T-shirts, we were quoted a total price
of $415. From this information, construct a linear function to model the cost of T-shirt orders. Use this model to calculate the cost of ordering 100 T-shirts, and determine how many T-shirts could be ordered for $1000.

**Answer 7:** The first step is to identify the independent and dependent variables. The related quantities are the cost of the T-shirt order and the size of the T-shirt order. Since the cost of the order depends on how many T-shirts are ordered, the size of the order will be the independent variable and the cost of the order will be the dependent variable.

Next we identify that there is a slope in the problem. For each unit (T-shirt) of the independent variable (size of the order), the dependent variable (cost of the order) increases by seven units (dollars). Our slope is seven. The y-intercept would be the fixed cost of setting up the machinery, which we do not know, but we do have data for one point in our function: when we order 50 T-shirts, the cost is $415. This gives the point (50, 415). Plugging this in the partial function \( f(x) = 7x + b \) gives a y-intercept of \( b = 65 \) (which corresponds to the fixed cost). Our function is \( f(x) = 7x + 65 \).

With this model, we can answer the last parts of the question. A T-shirt order of 100 would cost \( f(100) = 7 \times 100 + 65 = 765 \) dollars. To determine the number of T-shirts that could be ordered for $1000, find the value of \( x \) corresponding to \( f(x) = 1000 \). \( 1000 = 7x + 65 \) leads to \( 935 = 7x \), so \( x = 133.57 \). Since fractional shirts aren’t manufactured, this number should be rounded down to 133 T-shirts.

It should be noted that letters other than \( f \) and \( x \) could be used. The function \( f(x) = 7x + 65 \) could just as easily be recorded as \( P(t) = 7t + 65 \), where \( P \) is the price of the order and \( t \) is the number of T-shirts ordered.

**Question 8:** Irving has noticed that his ice cream sales increase when the day’s high temperature gets hotter. Yesterday, when the high temperature reached 82ºF he sold 50 ice cream cones. Today the high temperature reached 87ºF and he sold 62 ice cream cones. Construct a linear function to model Irving’s ice cream cone sales. Tomorrow’s high temperature is forecast to be 90ºF; use your model to predict how many cones he might sell.

**Answer 8:** The quantities in this problem are temperature and ice cream cone sales. Since the warmer temperatures drive customers to the ice cream stand and not the reverse (more people buying ice cream will not cause the temperature to increase), temperature will be the independent variable, represented by \( t \), and ice cream cone sales will be the dependent variable, represented by \( C \).

This problem does not offer a slope, but rather gives two data points, namely (82, 50) and (87, 62). Note that since we chose temperature as the independent variable, we list the temperatures first for the data points. We can use these two points to calculate a slope: \( m = \frac{62 - 50}{87 - 82} = \frac{12}{5} = 2.4 \). Next we plug our choice of the two points into the partial function \( C(t) = 2.4t + b \), which gives a y-intercept of \( b = -146.8 \). Our function is \( C(t) = 2.4t - 146.8 \). With this function, a high temperature of 90ºF might lead to ice cream cone sales of \( C(90) = 2.4 \times 90 - 146.8 = 216 - 146.8 = 69.2 \); about 69 ice cream cones.
It should be noted that not all relationships where two data points are gathered are accurately modeled with a linear function. In many cases where a linear function is used, there may be a limited range of values where the model applies, implying that the model should not be used beyond this range. In the example of our ice cream cone sales model, \( C(t) = 2.4t - 146.8 \), the model might apply when high temperatures are in the range from 75°F to 95°F, but the model would not be useful if the high temperature were only 50°F (implying negative ice cream cone sales since \( C(50) = 2.4 \times 50 - 146.8 = -26.8 \)) or if the high temperature were to reach 115°F (since temperatures this warm would keep many prospective customers at home).

**Exercises**

1. Determine the linear function with slope two containing the point \((0, 5)\).
2. Determine the linear function with slope three containing the point \((0, 1)\).
3. Determine the linear function with slope \(-4\) containing the point \((2, 3)\).
4. Determine the linear function with slope \(\frac{2}{3}\) containing the point \((6, -2)\).
5. Determine the linear function containing the points \((0, 2)\) and \((4, 4)\).
6. Determine the linear function containing the points \((0, 5)\) and \((3, 1)\).
7. Determine the linear function containing the points \((4, -3)\) and \((-2, 0)\).
8. Determine the linear function containing the points \((-3, -1)\) and \((1, 5)\).
9. Paulo’s Pizza will deliver any number of pizzas for only $6 each, but he also adds a $2 delivery charge, which is the same regardless of how many pizzas are being delivered.
   a. Model the cost of a pizza order as a linear function.
   b. Based on this model, how much will it cost to order nine pizzas?
   c. Based on this model, how many pizzas could be ordered for $110?
10. Rita’s Rent-A-Car has purchased a new hybrid for her fleet at a cost of $28,000. Each year she estimates the depreciation on that car to be about $3,500, meaning the car is losing value at a rate of $3,500 per year.
    a. Model the value of the car as a linear function.
    b. Based on this model, what will be the value of the car after five years?
    c. Based on this model, how long will it take for the car to be worth nothing?
11. Before a rainstorm began, suppose a barrel collecting gutter water was four inches deep. After an hour of heavy rain, the barrel was ten-and-a-half inches deep.
    a. Model the depth of the rainwater as a linear function.
    b. Based on this model, how deep will the water be after two more hours (a total of three hours) of heavy rain?
    c. Based on this model, how many hours of heavy rain (from the start of the storm) would it take to overflow the barrel if it is 50 inches high?
12. Tom is reading his history textbook for homework, and his assignment starts on page 27. After 20 minutes of reading, he notices he is on page 39.
   a. Model the page number Tom has reached as a linear function.
   b. Based on this model, what page will Tom be reading after a total of 50 minutes of reading?
   c. Based on this model, how long will it take Tom to complete his assignment, which ends on page 66?
13. Suppose a rubber band with a 100-gram weight hanging from it becomes ten centimeters long. The same rubber band with a 160-gram weight becomes 11.5 centimeters long.
   a. If we designate $r$ to be the length of the rubber band, construct a linear function $W(r)$, to predict the hanging weight when rubber band stretches to a length of $r$.
   b. Use this function to predict the weight of an object that stretches the rubber band to nine centimeters.
   c. Determine how long the rubber band is when it is at rest (with no weight hanging from it).

14. Crickets are cold-blooded creatures and so their body temperature will match the surrounding air temperature. As the temperature warms, so does the cricket, allowing the cricket to move faster and chirp faster. Val counted 152 cricket chirps in a minute on an evening when the temperature was 78°F and counted 120 cricket chirps in a minute the next morning when the temperature was 70°F.
   a. If we designate $c$ to be the number of cricket chirps in a minute, construct a linear function $T(c)$ to predict the temperature based on this data.
   b. Use this function to predict the temperature when she hears 132 chirps in a minute.
   c. Determine how many cricket chirps she would expect to hear in a minute when the temperature is 81°F.

Is It Reasonable?

15. Sebastian is grilling burgers at a family picnic. He noticed that it took 6 minutes to grill one burger on his grill. He uses a linear function to estimate that it will take 24 minutes to cook all the burgers while there are four burgers on the grill. Is this linear model reasonable?

16. Taylor is making copies of course notes for the students in her course. She was quoted a price of $4 to make one copy of her notes. She uses a linear function to estimate that 14 copies will cost around $56. Is this linear model reasonable?

17. Victor is driving to the Grand Canyon, which is about 2000 miles away. On the first day, Victor was able to travel about 500 miles. He uses this to estimate that it will take another three days to travel the remaining 1500 miles. Is this linear model reasonable?

18. Wystan finds that an eight-inch diameter pizza can feed one person. She concludes that a 16-inch diameter pizza should therefore feed two similar people. Is this linear model reasonable?

Advanced Applications

19. Suppose a barber can cut one person’s hair in 20 minutes.
   a. If we designate $h$ as the number of hours in a work day, construct a linear function $C(h)$ to predict the number of customers that can be served in the $h$-hour work day.
   b. If the barber refuses to start cutting anyone’s hair if there is not time to finish, then a work day where the function predicts 16.7 customers would only allow 16 customers to be served. Sketch a graph of the function modeling the number of customers that can be served taking this into account.

20. A long distance phone company has an offer where you can talk to anyone in Brazil or Australia for only two cents a minute, plus a 49-cent connection fee.
   a. Model the cost of a call using a linear function.
   b. When the phone company calculates the cost of a call, the lengths of the calls are rounded up to the next minute, so a call lasting 13 minutes and 18 seconds would cost the same as a call lasting 14 minutes. Sketch a graph of the function modeling the cost of the call taking into account this rounding.
1.7 Review Exercises

1. Determine and simplify the following expressions:
   a. A collection of 11 coins contains pennies and nickels. If \( p \) represents the number of pennies, how many nickels are in the collection (in terms of \( p \))?
   b. Given four consecutive integers, call the smallest one, \( x \). What is the expression that represents “two more than three times the largest integer” (in terms of \( x \))?

2. Solve the following equations:
   a. \( 7r - 3 = 2(3 - r) \)  
   b. \( \frac{1}{2}(5x + 8) + 3 = 15 - x \)  
   c. \( 3(9 - 4(q - 1) + 2) = 1 - q \)  
   d. Solve for \( g \): \( 3(c - 2 + g) = a - 3g \)

3. Solve the following problems:
   a. A collection of 11 coins containing pennies and nickels is worth 35¢. How many nickels are in the collection?
   b. Consider four consecutive integers. If the sum of all the integers is equal to two more than three times the largest integer, what are the integers?

4. Determine which of the following graphs are functions.

5. Let \( f(x) \) be the function defined by \( f(x) = 3x - 2 \). Calculate and simplify the following
   a. \( f(3) \)  
   b. \( f(-1) \)  
   c. \( f(\frac{5}{6}) \)  
   d. \( f(q) \)

6. Determine which of the following are equations of lines. For those that are lines, determine the slope and \( y \)-intercept.
   a. \( y = 3x + 7 \)  
   b. \( y = (x + 3)(x - 1) \)  
   c. \( y = (4 - x) + 4 \)  
   d. \( 4y - 3x = 0 \)  
   e. \( y = \frac{6}{x} + 4 \)  
   f. \( y = 9999x - 2000 \)

7. Find the equation for the following lines:
   a. The line containing the point (2, 2) with a slope of \( \frac{1}{2} \).
   b. The line containing the point (3, 5) with a slope of 2.
   c. The line containing the points (1, –1) and (0, 3).
   d. The line containing the points (4, 2) and (7, 6).

8. Math, the magazine, is just about to publish their first issue. It will cost a total of $300 to print 2000 copies. If 2500 copies are printed it will cost $340.
   a. Find a linear function which relates the cost of printing to the number of copies printed.
   b. Use this function to predict the cost of making 2800 copies.
   c. How many issues could be printed for $400?
1.8 Project: Linear Regression

Since measurements are never exact, it is sometimes necessary to make a guess at a linear model based on data points that do not lie all on the same line. Finding the best line through multiple points that do not form a perfect line is called linear regression. The formulas and theory are beyond the scope of this course, but many software packages provide the tools to calculate a linear regression for you.

The following project requires the use of Microsoft Excel®. Similar spreadsheet programs may be used, but the instructions may differ slightly.

A. Open Microsoft Excel® (or similar program)

B. In the left-most column, enter the x values for your points, and in the second column from the left, enter the corresponding y values for your points.

For example, if your points are (2,10), (5,8), (10,5), and (12,6), your spreadsheet should look like the first figure to the right:

C. Select the cells containing your data (drag from one corner to the other).

D. From the insert menu, insert a chart. Choose the “XY (Scatter)”. Select the figure with no lines, just points.

E. Click on any of the data points (this will select them all), and then right-click on any of the data points and choose “Add Trendline…”.

F. In the trendline options, choose “Linear” (it is probably the default choice, since it is the most common), and check the box next to “Display Equation on chart”.

The equation on the chart represents the best linear model that can fit the available data. If the points are randomly scattered, these trendlines will not be very useful, but if the data appears to be lined up, or correlated, these trendlines can be very useful. Correlation will be discussed more in chapter four.

Use the instructions above to do the following exercises:

1. Graph the following points in a scatter plot:
   \((1,2)\) \((2,5)\) \((3,4)\) \((4,6)\) \((5,11)\) \((6,11)\) \((7,19)\) \((8,20)\) \((9,16)\) \((10,21)\)
   Before you generate the trendline, try to predict (approximately) the slope and y-intercept. Then compare the true values generated by Excel to your prediction.

2. Graph the following points in a scatter plot:
   \((1.12)\) \((2.4)\) \((3.8)\) \((4.7)\) \((5.15)\) \((6.12)\) \((7.2)\) \((8.9)\) \((9.8)\) \((10,12)\)
   Again, before you generate the trendline, try to predict (approximately) the slope and y-intercept. Then compare the true values generated by Excel to your prediction. Why was this one harder to predict than the first?
3. Generate your own data points using a six-sided die by doing the following:

- Roll a six-sided die several times, and record on which rolls you get a six.
- Continue this until you have rolled ten sixes. (If it takes you more than 100 rolls, check the die you are using.)
- The \( x \) coordinates will be the numbers from one to ten, while the \( y \) coordinates will be the rolls on which you got a six.

For example, if in the first 25 rolls, you got a six on the 7\(^{th}\), 10\(^{th}\), 15\(^{th}\), and 23\(^{rd}\) rolls, then your first four data points would be (1,7), (2,10), (3,15), and (4,23).

Generate the scatter plot and calculate the trendline and trendline equation.

If you were to continue tracking points for the first 1000 rolls, what would you predict the slope and \( y \)-intercept of the trendline to be? Do you have any intuition as to why this should happen?
2 Using Numbers

2.1 Approximations

Measurements are never exact. The task of tracking the margin of error for calculations based on measurements can be extremely important. How much weight can this bridge support? How hot will this reactor get? Without an idea of how much error may be in these calculations, we cannot know where to set safety tolerances for safe operation.

Significant Figures

One simple concept for tracking accuracy in calculations is the idea of significant digits. If a distance is measured to be 360 feet, we can assume the true distance to be somewhere between 350 and 370 feet. The six in the number 360 is the last significant digit. Since this digit lies in the tens column, we assume an error of no more than ten units.

If a distance is measured to be 357 feet, we can assume the true distance to be somewhere between 356 and 358 feet. The seven in the number 357 is the last significant digit. Since this digit is in the ones column, we assume an error of no more than one unit.

If a distance is measured to be 356.2 feet, we can assume the true distance to be somewhere between 356.1 and 356.3 feet. The two in the number 356.2 is the last significant digit. Since this digit is in the one-tenths column, we assume an error of no more than one-tenth of a unit.

The number of significant figures in a number is the number of digits used when writing the number with two exceptions:

- Zeroes at the end of a number with no digits after a decimal point are not considered significant.
- Zeroes at the beginning of a number with no non-zero digits before a decimal point are not considered significant.

**Question 1:** How many significant figures in the measurement 724.6 feet and how big might the error be?

**Answer 1:** The number has four digits and none are leading zeroes or trailing zeroes, thus the number has four significant figures. The last significant digit is in the one-tenths column, so the error may be as big as 0.1 feet.

**Question 2:** How many significant figures in 94,300 pounds and how big might the error be?

**Answer 2:** The number has five digits, but the last two are trailing zeroes of a number with no digits after a decimal point, so the number has three significant figures. The last significant digit is in the hundreds column, so the error may be as big as 100 pounds.

**Question 3:** How many significant figures in 0.024 seconds and how big might the error be?

**Answer 3:** The number has four digits, but since there are no non-zero digits left of the decimal point, the leading zeroes are not significant, so the number has two significant figures. The last significant digit is in the one-thousandths column, so the error may be as big as 0.001 seconds.

**Question 4:** How many significant figures in 0.000230 meters and how big might the error be?
Answer 4: The number has seven digits, but again, the leading zeroes do not qualify. However, trailing zeroes after a decimal point do qualify, so this number has three significant figures. The last significant digit is in the one-millionths column, so the error may be as big as 0.000001 meters.

Question 5: How many significant figures in 100.00 gallons and how big might the error be?

Answer 5: The number has five digits, but you might think the trailing zeroes should not qualify. However, because there are digits after the decimal point, the trailing zeroes will qualify, so this number has five significant digits. The last significant digit is in the one-hundredths column, so the error may be as big as 0.01 gallons.

Multiplication and Division

When multiplying or dividing measured quantities, the number with the smaller number of significant figures determines the accuracy of the answer. Consider measuring the square-footage of a rectangular roof where one dimension is 308 feet and the other dimension is 42 feet. Simply multiplying the numbers gives a product of 12,936 square feet, but this answer implies five significant figures of accuracy and an error of no more than one square foot. Based on the measurements of the dimensions, the roof could be as small as 307 \times 41 = 12,587 square feet, or as big as 309 \times 43 = 13,287 square feet. This is a range of 700 square feet, far from the one square foot error implied by the answer 12,936. Examining the significant figures of the dimensions, 308 has three figures and 42 has two figures. Since two is the smaller number, we can only expect our answer to have two significant figures, so 12,936 must be rounded to 13,000.

Question 6: Calculate the average speed to the correct precision if a car travels 146 miles in 3.0 hours.

Answer 6: To calculate miles per hour, divide miles by hours. This would be \( \frac{146}{3.0} = 48.666666... \) miles per hour. Since 146 has three significant figures and 3.0 has two significant figures, our answer should be rounded to the smaller number of significant figures, two. We have 49 miles per hour.

Question 7: Calculate the number of kilowatt-hours consumed by an appliance requiring 128 kilowatts for 0.125 hours.

Answer 7: To calculate kilowatt-hours, multiply kilowatts by hours. This would be \( 128 \times 0.125 = 16 \) kilowatt-hours. Since both 128 and 0.125 have three significant figures, our answer should have three significant figures. Rather than writing 16 as the answer, it should be 16.0 kilowatt-hours.

Addition and Subtraction

When adding or subtracting measured quantities, the number whose last significant digit is furthest to the left determines the accuracy of the answer. Consider adding distances of 40 inches and 16.3 inches. Simply adding the numbers gives a sum of 56.3, but this answer implies accuracy to within one-tenth of an inch. While the second measurement, 16.3 inches, implies accuracy to within one-tenth, the first measurement, 40 inches, has only one significant figure and thus implies accuracy only to within ten inches. Adding 16.3 to this quantity does not make the first measurement more accurate, so we must express our answer to accuracy within ten inches, so 56.3 must be rounded to 60 inches.
**Question 7:** Estimate the weight in an elevator with a man weighing 75 kilograms and a package weighing 4.25 kilograms.

**Answer 7:** Adding 75 and 4.25 gives 79.25. The first measurement implies accuracy to within one kilogram, the second to within one one-hundredth of a kilogram. The first is the least accurate, and thus determines the accuracy of the answer. Rounding to the nearest kilogram gives 79 kilograms.

**Question 8:** Estimate the distance traveled by an astronaut from the moon to Cape Canaveral, which is 240,000 miles, and then from Cape Canaveral to Disney World, which is about 68 miles.

**Answer 8:** Adding gives 240,068 miles, but the accuracy of the first measurement is far worse than the second, implying an error up to 10,000 miles. The error on the first measurement dwarfs the entire second measurement, and rounding to the nearest 10,000 miles gives 240,000 miles.

**Exercises**

1. Consider the measurements of a rectangle: 120 meters, 6.55 meters.
   a. How many significant figures in each measurement?
   b. Determine the area by multiplying the measurements to the correct accuracy.

2. Consider the measurements for a moving car: 487.3 miles, 8.5 hours.
   a. How many significant figures in each measurement?
   b. Determine the average speed by dividing distance by time to the correct accuracy.

3. Consider the measurements of liquid ingredients: 96.0 ounces, 8.77 ounces.
   a. What is the accuracy of each measurement?
   b. Determine the total liquid by adding the measurements to the correct accuracy.

4. Consider the measurements of a wrestler two weeks apart: 168 pounds, 159.4 pounds.
   a. What is the accuracy of each measurement?
   b. Determine the weight lost by the wrestler by subtracting the measurements to the correct accuracy.

**Calculate each of the following to the correct accuracy.**

5. 1290 × 34.78

6. 0.0060 × 3572

7. \[
\frac{295}{0.01}
\]

8. \[
\frac{2468}{2.0}
\]

9. 68.7 + 170

10. 12.00 + 89.9

11. 1500 − 76.54

12. 24,523 − 300
Is It Reasonable?

13. Using a typical ruler, Greg measures the length of a pen as 15.333 centimeters. Is the implied accuracy of this measurement reasonable?

14. Using a typical bathroom scale, Helena measures the weight of her dog as 30 pounds. Is the implied accuracy of this measurement reasonable?

Advanced Applications

15. 93.80 – 93.796

16. 3.32 × 3.0
2.2 Scientific Notation

**Question 1:** Calculate 25 × 12 to the correct accuracy.

**Answer 1:** $25 \times 12 = 300$. Since there are two significant figures in each number, we need to have two significant figures in our answer. Unfortunately, there is no way to designate the first zero as significant while the second zero is not. Writing 300 implies just one significant figure.

Had question one been $2.5 \times 1.2$, this would not have been a problem. The product here is three, and designating two significant figures would be 3.0. If answers were always no less than one and always less than ten, demonstrating significant figures would never be a problem. *Scientific notation* allows this.

**Answer 1:** 300 in scientific notation would be three times one hundred, which is the second power of ten. In scientific notation this would be $3 \times 10^2$. Displaying two significant figures in the answer is now easily done, giving a final answer of $3.0 \times 10^2$.

One way to imagine the conversion to scientific notation is counting how many places the decimal point must be moved to make the number less than ten but not less than one. The decimal point must be moved two places to left to convert 300 into three, therefore three must be multiplied by ten twice to make it 300.

**Question 2:** Express 240,000 miles in scientific notation.

**Answer 2:** The decimal point needs to move five places, so we have $2.4 \times 10^5$ miles. Note that we assume the trailing zeros on the measurement are not significant.

**Question 3:** Express 0.00310 meters in scientific notation.

**Answer 3:** The decimal place is moving three places, but in this case, it is moving to the right. Multiplying 3.10 by one thousand ($10^3$) would give 3,100. To get 0.00310, 3.10 must be divided by one thousand ($10^3$). Division is represented by a negative exponent, so the answer would be $3.10 \times 10^{-3}$ meters.

**Question 4:** Express $1.28 \times 10^7$ people (population of Pennsylvania) in standard notation.

**Answer 4:** $10^7$ is ten million. Multiply this by 1.28 to get 12,800,000. Notice that there are only five zeroes at the end of this number. The first two moves of the decimal point did not generate a zero at the end of the number.

**Question 5:** Express $1.0 \times 10^{-10}$ meters (width of a hydrogen atom) in standard notation.

**Answer 5:** $10^{-10}$ is one divided by ten billion. Multiply this by 1.0 and we get 0.0000000001.0. Notice there are only nine zeroes between the decimal point and the one. The first move of the decimal point did not generate a leading zero.

As exponents get bigger, we can see another advantage of scientific notation. The human eye gets tired trying to count more than a few leading or trailing zeroes.
**Question 6:** Multiply \(3.92 \times 10^5\) by \(1.4 \times 10^7\).

One approach would simply be to convert the measurements to regular notation, perform the calculation, and then convert the number back to scientific notation. However, when exponents are large, tracking the leading or trailing zeroes becomes tiring. There is a better way.

**Answer 6:** Rearrange the multiplication as \((3.92 \times 1.4) \times (10^5 \times 10^7)\). The first multiplication gives 3.92 \times 1.4 = 5.488, which should be rounded to 5.5 since we should have only two significant figures in our answer. The second multiplication gives \(10^5 \times 10^7 = 10^{12}\). Notice that we add the exponents since we are multiplying five 10’s by seven 10’s for a total of twelve 10’s being multiplied together. Our final answer: 5.5 \times 10^{12}.

**Question 7:** Calculate \(9.38 \times 8.42 \times 10^2 \times 10^{10}\) (number of atoms in four metric tons of hydrogen).

**Answer 7:** Rearrange the division as \(\frac{8.423 \times 10^9}{2 \times 10^5}\). The first division results in 4.2115, which should be rounded to 4 since we should only have one significant figure. The second division results in \(10^6\). Notice that we subtract exponents since we are dividing nine 10’s by three 10’s, leaving six 10’s being multiplied together. Our final answer: 4 \times 10^6.

**Question 8:** Multiply \(4.000 \times 10^6\) by \(6.02 \times 10^{23}\) (number of inches in flight from Los Angeles to Hartford through New York).

**Answer 8:** Rearrange to \((4.000 \times 6.02) \times (10^6 \times 10^{23})\). Multiplying gives 24.1 \times 10^{29} (notice we rounded to three significant figures). However, we are not done since 24.1 is not less than 10. We need to move the decimal point one more place, to get 2.41, in order to put this into scientific notation. This requires that we adjust the power of ten by one as well. Our final answer: 2.41 \times 10^{30} (that’s 2.41 nonillion).

**Question 9:** Calculate \(\frac{2.7 \times 10^3}{3.0 \times 10^8}\) (number of seconds needed for light to cross the Golden Gate Bridge).

**Answer 9:** Rearrange the division as \(\frac{2.7}{3.0} \times \frac{10^3}{10^8}\), which gives 0.90 \times 10^{-5}. Notice we need two significant figures in our answer and that subtracting the exponents gave a negative number. Unfortunately, 0.90 is less than one, so we need to move the decimal point. Since 0.90 becomes smaller when converted to 2.41, to compensate, the power of ten must get larger, so 24.1 \times 10^{29} becomes 2.41 \times 10^{30}.

**Question 10:** Add \(1.55 \times 10^8\) and \(6.1 \times 10^6\) (number of inches in flight from Los Angeles to Hartford through New York).

**Answer 10:** Adding or subtracting scientific notation requires the exponents to agree, so one of the numbers must be converted. It does not matter which one is converted, but converting the smaller number will often translate to fewer steps (not always). To make the exponent in \(6.1 \times 10^6\) equal to eight, the exponent must get larger by two. To compensate, the decimal place in 6.1 must move two places to make
the number smaller, so $6.1 \times 10^6 = 0.061 \times 10^8$. Now we can add: $1.55 \times 10^8 + 0.061 \times 10^8 = 1.611 \times 10^8$. However, since 1.55 is accurate to only one hundredth, the answer must be rounded to $1.61 \times 10^8$.

**Question 11:** Subtract $9.86 \times 10^1$ from $1.000 \times 10^2$.

**Answer 11:** To calculate $(1.000 \times 10^2) - (9.86 \times 10^1)$, first convert the exponent on one of the numbers to make the exponents match. Converting the first, we get $9.86 \times 10^1 = 0.986 \times 10^2$. Then subtract: $1.000 - 0.986 = 0.014$, so we have $0.014 \times 10^2$. Both numbers were accurate to the thousandths place, so no rounding is needed. Finally, we need to move the decimal point two places to convert $0.014$ into $1.4$, which makes the number larger. To compensate, we must reduce the exponent by 2, giving $1.4 \times 10^0$. Since $10^0 = 1$, this could simply be written as $1.4$. Some science teachers will maintain that without including $\times 10^0$ it is not really scientific notation.

**Question 12:** Subtract $9.86 \times 10^1$ from $1.0 \times 10^2$.

**Answer 12:** This question starts the same: $9.86 \times 10^1 = 0.986 \times 10^2$. Then subtract: $1.0 - 0.986 = 0.014$. However, 1.0 only gives accuracy to the tenths, so 0.014 must be rounded to 0.0, which looks a little strange. At this point, we realize that no moving of the decimal point will make 0.0 less than ten but not less than one. Our answer is zero, which cannot be expressed in scientific notation.

**Orders of Magnitude**

In some cases, we may be interested in less than one significant figure. Simply knowing the power of ten for a number gives an idea of how large a number is. The power of ten for a number is called the order of magnitude of the quantity. Is it in the millions, ten-thousands, or hundred-billions? This is done by rounding a number to its nearest power of ten. Calculating with these simple quantities is often called a Fermi Estimation, named after physicist Enrico Fermi. When rounding a number to its nearest power of ten, round down if the number in front of the power of ten is less than 3.2, and round up if the number is 3.2 or greater.

**Question 11:** Estimate how hours of American network-broadcast prime-time television there are.

**Answer 11:** Since we are looking for a very rough estimate, round each number in your calculation to the nearest power of 10. Prime time television on each network lasts about three hours. Three rounded to the nearest power of ten is 1. For most of the time television has been operating, there have been anywhere from 4 to 7 networks. Rounding to the nearest power of 10 gives 10 networks. Prime-time network broadcasts started shortly after World War II. Rounding to the nearest power of 10 means each network has generated about 100 seasons of prime-time television. Finally, each season of programming usually has 20 to 30 episodes, which to the nearest power of 10 is about 10 episodes. Multiplying these numbers together gives a total of $1 \times 10 \times 100 \times 10 = 10,000$ hours of broadcast prime-time television. This is a very rough approximation, and does not include daytime television or cable television series.

**Question 12:** Estimate how many ounces of coffee are consumed in the United States each year?

**Why 3.2?**

(Actually, 3.162277660…)

Note that when the addition rule for multiplying powers of ten is applied to halves, we get $10^{1/2} \times 10^{1/2} = 10^1$. This means that $10^{1/2}$ must be the square root of ten, about 3.2, and a number larger than this should round up to the next power of ten.
Answer 12: Again, we wish to approximate quantities in the problem to the nearest powers of 10. Each cup of coffee probably has eight to 16 ounces, which rounds to 10 ounces. The population of the United States is about 300 million, including children. Assuming half of the population drinks coffee regularly, that is 150 million, which rounds to 100 million or $10^8$. Some people drink multiple cups a day, others drink less than a cup per day, but suppose it averages out to about one cup per day. Finally, a year has 365 days, which, to the nearest power of 10 rounds to 1000 or $10^3$. Multiplying these quantities gives a grand total of $10^4 \times 10^8 \times 1 \times 10^3 = 10^{12} = \text{one trillion ounces.}$

Exercises

1. Write the following numbers in standard notation:
   a. $6.62 \times 10^3$
   b. $3.1 \times 10^{-4}$
   c. $137 \times 10^{10}$

2. Write the following numbers in standard notation:
   a. $7.2 \times 10^5$
   b. $1.67 \times 10^{-3}$
   c. $5555 \times 10^{8}$

3. Write the following numbers in scientific notation:
   a. 350,000
   b. 0.000087
   c. 135.70
   d. $98.2 \times 10^{-35}$

4. Write the following numbers in scientific notation:
   a. 2,300,000
   b. 0.00032
   c. 2400.0
   d. $0.86 \times 10^{30}$

Calculate the following in scientific notation to the correct level of accuracy:

5. $(2.34 \times 10^5) \times (3.7 \times 10^4)$

6. $(5.482 \times 10^7) \times (1.21 \times 10^8)$

7. $(2.900 \times 10^8) \times (5.61 \times 10^{-5})$

8. $(3.8 \times 10^{-6}) \times (9 \times 10^3)$

9. $\frac{1.73 \times 10^6}{3.791 \times 10^4}$

10. $\frac{5.9 \times 10^3}{2.95 \times 10^{-5}}$

11. $(2.34 \times 10^5) + (3.7 \times 10^4)$

12. $(5.482 \times 10^7) + (1.21 \times 10^8)$

13. $(2.900 \times 10^8) - (5.61 \times 10^5)$

14. $(9 \times 10^5) - (3.8 \times 10^{-6})$
Is It Reasonable?

15. Which is a more reasonable estimate for the number of popcorn kernels eaten in American movie theaters in one year: $10^5$ or $10^{10}$?

16. Which is a more reasonable estimate for the number of shots taken at a popular miniature golf course in one day: $10^4$ or $10^9$?

Advanced Applications

17. Calculate to the correct precision in scientific notation: $[(9.96 \times 10^2) - (1.22 \times 10^3)] \times (4.54 \times 10^{-3})$

18. Calculate to the correct precision in scientific notation: $\frac{(7.0312 \times 10^5) - (7.1309 \times 10^5)}{1.0 \times 10^2}$
2.3 Converting Units

Calculations involving unit conversion are quite common, whether you need to convert centimeters to inches to see if your bookcase will fit through a doorway, or convert Dollars to Euros in order to see how much you really are spending on that T-shirt. The most common confusion is trying to determine when one should multiply or divide. One simple guiding principle should make these calculations straightforward, no matter what kind of unit you are converting:

Keep the Units

Unit conversion calculations become much less of a mystery when you add the units to the numbers. Working with numbers alone will often work well when the calculation involves a single conversion and we know which unit is larger, but if you are converting multiple units—miles per gallon to kilometers per liter, for example—keeping the units attached to the number will make the correct approach obvious.

**Question 1:** Convert 136 feet to inches.

This is a simple example, but will serve to illustrate the concept of keeping the units.

**Answer 1:** The conversion needed is 1 foot = 12 inches. This can be expressed as a ratio of equal measure when written as $\frac{1 \text{ ft}}{12 \text{ in}}$ or $\frac{12 \text{ in}}{1 \text{ ft}}$. Someone who knows that feet are larger than inches would instinctively reach for the second ratio to use in the conversion, but even without this knowledge, examining the two possibilities leads to one obvious conclusion. Calculating $136 \text{ ft} \times \frac{1 \text{ ft}}{12 \text{ in}}$ leads to units of the form $\frac{\text{ft}^2}{\text{in}}$, which is not what we are after. Calculating $136 \text{ ft} \times \frac{12 \text{ in}}{1 \text{ ft}}$ leads to units of the form $\frac{\text{ft} \times \text{in}}{\text{ft}}$, which leaves only inches when the feet are divided out. Multiplying 135 by 12 gives 1632. Since the unit conversion of 12 is exact, it should be treated as 12.000000… when considering significant figures and accuracy. 135 has three significant figures and thus so should our answer: 1630 inches.

**Question 2:** Convert 100 kilometers to miles.

**Answer 2:** The conversion is 1 mile = 1.609 kilometers, which can be expressed as a ratio of equal measure as $\frac{1 \text{ mi}}{1.609 \text{ km}}$ or $\frac{1.609 \text{ km}}{1 \text{ mi}}$. Calculating with the second ratio leads to units of $\frac{\text{km}^2}{\text{mi}}$, but the first ratio gives 100 km $\times \frac{1 \text{ mi}}{1.609 \text{ km}} = 62.1504$ miles, since the kilometer units divide out nicely. If the context for this problem states that 100 kilometers is a measured quantity, we only have one significant figure and should round our answer to 60 miles. If the context is a road sign stating that the distance to Toronto is 100 kilometers, it is likely that all three figures are significant, and we should round our answer to 62.2 miles. If the context is a speed limit sign on the Queen Elizabeth Way, then 100 is exactly the number of kilometers you may legally travel in one hour, and it is the four significant figures in the approximate conversion that determines that our answer should be rounded to 62.15 miles.
Question 3: How many seconds in one year?

Answer 3: In this case, there is no direct unit conversion, we must convert from years to days, from days to hours, from hours to minutes, and then minutes to seconds. The units in each ratio should multiply in such a way as to yield a single type of unit at each step.

\[
1 \text{ yr} \times \frac{365 \text{ days}}{1 \text{ yr}} \times \frac{24 \text{ hr}}{1 \text{ day}} \times \frac{60 \text{ min}}{1 \text{ hr}} \times \frac{60 \text{ s}}{1 \text{ min}} = 31,536,000 \text{ seconds.}
\]

The number “one” spelled out, does not specify how many significant figures. Usually, numbers spelled out are exact quantities, so our answer does not need to be rounded.

Question 4: Convert 65 miles per hour into feet per second.

Answer 4: The word “per” implies division, so we are converting \(\frac{65 \text{ mi}}{\text{hr}}\) into \(\frac{\text{ft}}{\text{s}}\). As long as we carefully apply our ratios, the units will guide us to a correct answer. Converting miles to feet, we need \(\frac{5280 \text{ ft}}{1 \text{ mi}}\); the miles in the denominator here will divide with the miles in \(\frac{65 \text{ mi}}{\text{hr}}\). To convert hours into seconds, we need \(\frac{60 \text{ min}}{1 \text{ hr}} \times \frac{60 \text{ s}}{1 \text{ min}}\). However, the units in \(\frac{65 \text{ mi}}{\text{hr}}\) have the hours in the denominator. The hours will only divide out if we have hours in the numerator in our conversion ratios, so we should use \(\frac{1 \text{ hr}}{60 \text{ min}} \times \frac{1 \text{ min}}{60 \text{ s}}\). Our calculation becomes \(65 \text{ mi} \times \frac{5280 \text{ ft}}{1 \text{ mi}} \times \frac{1 \text{ hr}}{60 \text{ min}} \times \frac{1 \text{ min}}{60 \text{ s}} = 95.3333 \frac{\text{ft}}{\text{s}}\). With two significant figures in \(\frac{65 \text{ mi}}{\text{hr}}\) and all conversions being exact, it is appropriate to round to \(95 \frac{\text{ft}}{\text{s}}\).

Question 5: Convert £5.38 (British Pounds) to US Dollars.

Answer 5: Since one US Dollar can buy 0.640 British Pounds (£), the conversion ratio we need is \(\frac{1 \text{ US Dollar}}{0.640 \text{ British Pounds}}\), so \(5.38 \times \frac{1 \text{ US Dollar}}{0.640 \text{ British Pounds}} = 8.41\), when rounded to three significant figures.

Question 6: Convert one Euro to Japanese Yen.

Answer 6: In this case, we have no direct conversion available; we must translate through US Dollars. Starting from 1€, we need \(\frac{1 \text{ US Dollar}}{0.911 \text{ Euros}}\) and \(\frac{123.9 \text{ Japanese Yen}}{1 \text{ US Dollar}}\) to ensure that the units for Euros and Dollars divide out, leaving only Yen. Since the amount “one” is spelled out, we assume this is an exact quantity, so the three significant figures in the Euro conversion determine the answer must have three significant figures. Multiplying these conversions yields ¥136 when rounded to three significant figures.

Question 7: Convert 4.3 square yards to square feet.

Answer 7: Converting yards to feet would use the conversion \(\frac{3 \text{ ft}}{1 \text{ yd}}\). However, since we are converting square yards \((\text{yd}^2)\) to square feet \((\text{ft}^2)\), we need to divide out units of \(\text{yd}^2\), not just yd. This requires multiplying the conversion ratio twice (in other words, squaring it): \(\frac{3 \text{ ft}}{1 \text{ yd}} \times \frac{3 \text{ ft}}{1 \text{ yd}} = \left(\frac{3 \text{ ft}}{1 \text{ yd}}\right)^2 = \frac{9 \text{ ft}^2}{1 \text{ yd}^2}\). From this we conclude we have exactly nine

\[
1 \text{yd}^2 = 9\text{ft}^2
\]
square feet in one square yard, and so 4.3 yd$^2$ would be about $4.3 \text{ yd}^2 \times \frac{9 \text{ ft}^2}{1 \text{ yd}^2} = 39 \text{ ft}^2$.

**Question 8:** Convert 25 cubic centimeters to cubic inches.

**Answer 8:** Converting centimeters to inches uses $\frac{1 \text{ in}}{2.54 \text{ cm}}$, but since we are converting cm$^3$ to in$^3$, we must cube the conversion. $25 \text{ cm}^3 \times \left(\frac{1 \text{ in}}{2.54 \text{ cm}}\right)^3 = 1.5256 \text{ in}^3$, which should be rounded to 1.5 in$^3$ since the original measurement had two significant figures.

**Question 9:** Which is heavier, 234 pounds or 112 kilograms?

**Answer 9:** The only way to answer this is to convert one amount to the units of the other; it does not matter which. Converting the second, we get $112 \text{ kg} \times \frac{1 \text{ lb}}{0.454 \text{ kg}} = 246.7 \text{ lbs}$. This rounds to 247 pounds, which is more than 234 pounds, so 112 kilograms is the heavier quantity.

**Question 10:** Which is more, 57.00 quarts or 54.00 liters?

Significant figures can be very important in problems like this.

**Answer 10:** Converting 57 quarts to liters generates a calculation of $57.00 \text{ qt} \times \frac{1 \text{ gal}}{4 \text{ qt}} \times \frac{3.79 \text{ L}}{1 \text{ gal}} = 54.0075 \text{ L}$, which is slightly more than 54.00 liters. However, the approximate conversion rate has only three significant figures, so 54.0075 liters must be rounded to 54.0 liters. Without a more precise conversion, we cannot be sure which is larger.

**Answer 10:** (with a more precise conversion) One gallon is 3.7854 liters when calculated more precisely, and redoing the calculation yields: $57.00 \text{ qt} \times \frac{1 \text{ gal}}{4 \text{ qt}} \times \frac{3.7854 \text{ L}}{1 \text{ gal}} = 53.94 \text{ L}$. 54.00 liters is the larger quantity.

**Exercises**

1. Convert 13.2 pounds to ounces.
2. Convert 1.2 cups to fluid ounces.
3. Convert a standard cell phone plan of 1500 minutes to hours.
4. Convert the approximate elevation of Mt. Everest, 29,000 feet, to miles.
5. Convert a rate of glacial flow, 45 feet per year, to inches per day.
6. Convert the gas mileage of a hybrid automobile, 42 miles per gallon, to kilometers per liter.
7. Convert the area of Alaska, 1,720,000 square kilometers, to square miles.
8. Convert the square footage of a typical one-family home, 2400 square feet, to square meters.

9. Convert the pressure exerted by 9.2 kilograms per square centimeter to pounds per square inch.

10. Convert the density of gold, 19.3 grams per cubic centimeter, to ounces per cubic inch.

11. Convert the cost of a $5.00 sandwich to Mexican Pesos.

12. Convert the cost of going to the top of the Eiffel Tower, 15.00€, to US Dollars.

13. Which is longer: 505 feet, or 155 meters?

14. Which is heavier: 8.6 ounces or 225 grams?

15. Which is the better gas mileage: 35 miles per gallon, or 14 kilometers per liter?

16. Which fuel is more expensive: $2.89 (US dollars) per gallon, or $1.15 (Canadian dollars) per liter?

Is It Reasonable?

17. ¥2500 is a reasonable price for which item: a candy bar, a tourist’s t-shirt, a one-night stay at a hotel, or a new car?

18. 20 Mexican pesos is a reasonable price for which item: a candy bar, a tourist’s t-shirt, a one-night stay at a hotel, or a new car?

19. 80 kilograms is a reasonable weight for which item: a cell phone, a gallon of water, a professional soccer player, or a cement truck?

20. 2,500,000 quarts is a reasonable estimate to fill which item: a typical bathtub, a typical dumpster, an Olympic swimming pool, or a ten story office building.

Advanced Applications

21. The speed of light is about $2.998 \times 10^8$ meters per second. One light-year is the distance light can travel in one year. The Andromeda galaxy is about 2.5 million light-years away from Earth. Convert this distance to meters and express the result in scientific notation.

22. Convert one million US dollars to Japanese Yen, then convert this to Euros using the result from Question 6 in this section of the text, and then convert this back to US dollars. If these conversion rates were exact quantities offered by a currency exchange business, how much profit would you make (round to nearest dollar)? (Currency exchange businesses generally include a service charge equal to a small percentage of the currency exchanged, so it is nearly impossible to make a profit by capitalizing on round-off error in exchange rates)
2.4 Metric Units

The standard units in science are metric, where conversions are generally as easy as multiplying by a power of ten. One kilometer is 1000 meters, and one meter is 100 centimeters. Once the common prefixes are learned, the conversion is the same regardless of the unit. One liter is 1000 milliliters, one meter is 1000 millimeters, one gram is 1000 milligrams, and one second is 1000 milliseconds. The abbreviations for the prefixes are the same regardless of the unit and are placed in front of the abbreviation for the unit they are modifying. A centimeter is abbreviated cm, while a milliliter is abbreviated mL.

**Question 1:** Convert 3.2 kilograms to grams.

**Answer 1:** Using the conversion of one kilogram to 1000 grams, we get a conversion ratio of \(\frac{1000 \text{ g}}{1 \text{ kg}}\). This leads to a calculation of \(3.2 \text{ kg} \times \frac{1000 \text{ g}}{1 \text{ kg}} = 3200 \text{ g}\).

**Question 2:** Convert 460 centimeters to meters.

**Answer 2:** Knowing that one meter is 100 centimeters, we calculate \(460 \text{ cm} \times \frac{1 \text{ m}}{100 \text{ cm}} = 4.6 \text{ m}\).

**Question 3:** Convert 3.8 liters to milliliters.

**Answer 3:** Since one liter is 1000 milliliters, we have \(3.8 \text{ L} \times \frac{1000 \text{ mL}}{1 \text{ L}} = 3800 \text{ mL}\).

**Question 4:** Convert 0.023 kilometers to centimeters.

**Answer 4:** It is safest to do this in two steps, converting first to meters, and then to centimeters. The calculation becomes: \(0.023 \text{ km} \times \frac{1000 \text{ m}}{1 \text{ km}} \times \frac{100 \text{ cm}}{1 \text{ m}} = 2300 \text{ cm}\).

Using the principle of keeping the units should prevent us from dividing where we should multiply and multiplying where we should divide.

**Question 5:** Convert 0.56 gallons to milliliters.

From the last section, we have a conversion for gallons to liters and a conversion for fluid ounces to milliliters. This means we may convert the gallons to fluid ounces and then to milliliters, or we can convert the gallons to liters and then to milliliters. Since the conversions of metric units are easier (just being powers of ten), it is likely easier to convert gallons to liters, and then to milliliters.

**Answer 5:** Our calculation will look like: \(0.56 \text{ gal} \times \frac{3.79 \text{ L}}{1 \text{ gal}} \times \frac{1000 \text{ mL}}{1 \text{ L}} = 2122.4 \text{ mL}\), which should be rounded to two significant figures: \(2100 \text{ mL}\).

**Question 6:** Convert 150 yards to millimeters.
Answer 6: None of the distance conversions in the last section between metric and non-metric use either yards or millimeters, so this is three steps: 150 yd × \( \frac{3 \text{ ft}}{1 \text{ yd}} \times \frac{1 \text{ m}}{3.28 \text{ ft}} \times \frac{1000 \text{ mm}}{1 \text{ m}} = 137,195 \text{ mm} \), which becomes 140,000 mm when rounded to two significant figures.

Note that it is best to use a conversion that has more accuracy than your measurement. The conversion from meters to feet has three significant figures while the measurement 150 yards has only two. If the measurement had been 150.000 yards, we should use more accurate conversions wherever possible.

Converting inches to centimeters is exact, so we could use a calculation like:

150.000 yd × \( \frac{3 \text{ ft}}{1 \text{ yd}} \times \frac{12 \text{ in}}{1 \text{ ft}} \times \frac{2.54 \text{ cm}}{1 \text{ in}} \times \frac{1 \text{ m}}{100 \text{ cm}} \times \frac{1000 \text{ mm}}{1 \text{ m}} = 137,160 \text{ mm} \)

Scientific Notation

Since conversions of metric units are powers of ten, calculating with scientific notation is quite natural. Recall that multiplying or dividing by powers of ten requires adding or subtracting exponents. Since mega-units use M and milli-units use m, the Greek letter μ (pronounced “myoo”) is used for micro-units.

Question 7: It takes \( 3.34 \times 10^{-9} \) seconds for light to travel one meter in a vacuum. Convert this to microseconds.

Answer 7: One microsecond is \( 10^{-6} \) seconds, so \( 3.34 \times 10^{-9} \text{ s} \times \frac{1 \mu \text{s}}{10^{-6} \text{ s}} = 3.34 \times 10^{-3} \mu \text{s} \). Be careful when subtracting negative exponents: \((-9) - (-6) = -9 + 6 = -3\).

Question 8: The sun generates power of about \( 3.9 \times 10^{14} \) terawatts. Convert this quantity to watts.

Answer 8: One terawatt is \( 10^{12} \) watts (abbreviated w), so \( 3.9 \times 10^{14} \text{ Tw} \times \frac{10^{12} \text{ w}}{1 \text{ Tw}} = 3.9 \times 10^{26} \text{ w} \).

Temperature

Conversion between degrees Fahrenheit (°F) and degrees Celsius (°C) is messier than most conversions because, unlike all the other conversions, zero in one scale does not equal zero in the other. For any other conversion, we know zero pounds are zero kilograms and zero gallons are zero milliliters; zero is zero. This is not true for the Fahrenheit and Celsius temperature scales.

What we do know is that under standard atmospheric pressure, water freezes at 32°F = 0°C, and water boils at 212°F = 100°C. We can use these two points of equality on the temperature scales to generate a linear function that will convert one temperature scale to the other.

If we make Celsius degrees the independent variable, our data points become (0, 32) and (100, 212). The slope of the line is \( \frac{212-32}{100-0} = 1.8 \), or if you like fractions, \( \frac{9}{5} \). The y-intercept

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is taken from the point (0, 32), thus we have a line equation of \( y = 1.8x + 32 \). Letting \( F \) be degrees Fahrenheit and \( C \) be degrees Celsius, this can be written as the more familiar: \( F = \frac{9}{5} C + 32 \).

We could similarly create a line to convert degrees Fahrenheit to degrees Celsius, or we can solve our existing line equation for the variable \( C \) by first subtracting 32 from both sides and then multiplying both sides by \( \frac{5}{9} \). Notice that multiplying by \( \frac{5}{9} \) effectively undoes the multiplication by \( \frac{9}{5} \). This equation could be put into standard form as \( C = \frac{5}{9} (F - 32) \), but \( \frac{160}{9} \) is a strange fraction and has a decimal that is not any friendlier: 17.7777...

**Question 9:** Convert 68 degrees Fahrenheit to degrees Celsius.

**Answer 9:** Plugging \( F = 68 \) into \( C = \frac{5}{9} (F - 32) \) and then calculating, we get \( C = \frac{5}{9} (68 - 32) = \frac{5}{9} (36) = 20 \), so 68°F = 20°C.

**Question 10:** Convert –40 degrees Celsius to degrees Fahrenheit.

**Answer 10:** Putting \( C = -40 \) into \( F = \frac{9}{5} C + 32 \) and then calculating gives \( F = \frac{9}{5} (-40) + 32 = -72 + 32 = -40 \). Therefore –40°C = –40°F, and –40 is the place where the two temperature scales agree.

**Exercises**

1. Convert 8 milliseconds (typical daylight shutter speed for a camera) to seconds.
2. Convert 2400 milligrams (maximum recommended sodium consumption per day) to grams.
3. Convert 42.195 meters (length of a marathon) to kilometers.
4. Convert 0.355 liters (size of a typical soft drink can) to milliliters.
5. Convert 1800 feet (height of the CN tower in Toronto) to centimeters.
6. Convert 0.17 pounds (weight of a typical bagel) to grams.
7. Convert 120 yards (length of an American football field including endzones) to kilometers.
8. Convert 6.0 quarts (typical amount of blood in an adult) to milliliters.
9. Convert \( 6 \times 10^{-10} \) seconds (time for one clock cycle in a personal computer) to microseconds.
10. Convert \( 4.5 \times 10^{12} \) meters (distance to Neptune) to kilometers.
11. Convert \( 6.0 \times 10^{24} \) kilograms (mass of the earth) to milligrams.
12. Convert \( 2 \times 10^{-5} \) meters (width of a human hair) to centimeters.
13. Convert 105 degrees Fahrenheit (average July high temperature in Phoenix) to degrees Celsius.

15. Convert –11 degrees Celsius (coldest recorded temperature in Hawaii) to degrees Fahrenheit.

16. Convert 420 degrees Celsius (melting point for zinc) to degrees Fahrenheit.

Is It Reasonable?

17. Which units would be the most reasonable to measure the weight of a hamburger patty: milligrams, grams, or kilograms?

18. Which units would be the most reasonable to measure the width of your thumb: centimeters, meters, or kilometers?

19. Which units would be the most reasonable to measure the duration of a human blink: nanoseconds, milliseconds, or seconds?

20. Which units would be the most reasonable to measure the volume of a car’s gas tank: microliters, milliliters, or liters?

Advanced Applications

21. Kelvin degrees are the same size as Celsius degrees, but the Kelvin scale reaches zero at a temperature of absolute zero, which is the complete absence of all heat. This is –273.15 degrees Celsius. Liquid Nitrogen boils at about 77 degrees Kelvin. Convert this to degrees Fahrenheit.

22. Computer memory has adopted metric prefixes, though these are not exact since computer storage is built in powers of two not powers of ten. One byte is the amount of memory required to store one character of a text file or one pixel of an image in the .gif format. One “kilobyte” is not \(10^3 = 1000\) bytes, but \(2^{10} = 1024\) bytes. One “megabyte” is not \(10^6\) bytes, but \(2^{20} = 1,048,576\) bytes. Every metric increase of three powers of ten is actually an increase of ten powers of two when referring to computer memory. Convert the capacity of an eight gigabyte hard drive to bytes exactly.
2.5 Percentages

Percentages are simply fractions with implied denominators of 100; thus the name “per cent.” 50% of the population of France is identical to $\frac{50}{100}$ of the population of France, and 7.3% of the federal budget is identical to $\frac{7.3}{100}$ of the federal budget.

Absolute Percentages

50% of the population of France and 7.3% of the federal budget are both examples of absolute percentages. Absolute percentages can often be recognized by the multiplication implied by the word “of”. To calculate an amount from a phrase with an absolute percentage, multiply the percentage, as a fraction or decimal, by the total given, which from this point will be called the reference value.

**Question 1:** Calculate 50% of the population of France (about 60 million people).

**Answer 1:** Multiply the reference value, in this case the population of France at about 60 million, by $\frac{50}{100}$, or 0.50 if you prefer decimals. The answer would be $0.50 \times 60 \text{ million} = 30 \text{ million}$.

**Question 2:** Calculate 7.3% of the federal budget ($3.0 trillion).

**Answer 2:** Multiplying the reference value of $3.0 \text{ trillion}$ by $\frac{7.3}{100}$, which as a decimal would be 0.073, we get $0.073 \times 3,000,000,000,000 = 219,000,000,000$. Rounded to two significant figures this is $220,000,000,000$ or $220 \text{ billion}$.

To calculate an absolute percentage, we must determine the quantity that we multiply by the reference value to get the final value. Algebraically, this looks like:

$$\frac{\text{absolute percentage}}{100} \times \text{reference value} = \text{final value} \quad \Rightarrow \quad \text{absolute percentage} = \frac{\text{final value}}{\text{reference value}} \times 100$$

There may be a little detective work needed to determine which value is the reference value and which is the final value. Examining the context carefully should help.

**Question 3:** In June of 2009, Philadelphia had measurable rainfall on 16 of the 30 days in the month. What percentage of days in June of 2009 did Philadelphia receive measurable rainfall?

**Answer 3:** The reference value is the total number of days in June, 30. The context gives this away looking at the placement of the word “of” in “percentage of days in June of 2009”. The final value is the other quantity, the number of days receiving measurable rainfall, 16. The percentage is the final value divided by the reference value, which is $\frac{16}{30} = 0.5333$. To write this number as a “per cent”, multiply by 100 and round to the appropriate decimal place. In this case, 16 and 30 are both exact amounts, so technically, we should give the exact percentage: 53.3333333333333…%. In cases where you have exact amounts with integer values, rounding to the nearest per cent is generally sufficient: 53% in this case.

**Question 4:** In June of 2009, Philadelphia received 4.46 inches of rainfall. Normally, Philadelphia receives only 3.29 inches of rainfall in June. What percentage of the normal rainfall did Philadelphia receive in June of 2009?

**Answer 4:** Once again, context determines that this is an absolute percentage, since we have the phrase “percentage of”. The reference value, which follows the “of”, is the normal rainfall Philadelphia receives in June, 3.29 inches. The final value must be the 4.46 inches Philadelphia got in 2009, making our percentage: $\frac{4.46}{3.29} = 1.3556$, which is 136% when expressed as a percentage rounded to three significant figures.
Notice that percentages are not always in the range of 0% to 100%. There are contexts where percentages greater than 100% are not possible, but this example shows that there are cases where percentages greater than 100% have a valid meaning.

Relative Percentages

When comparing rainfall amounts to normal, most weathercasters do not give a percentage of normal rainfall. Instead, they will give a percentage more than or less than normal rainfall totals. These phrases indicate a relative percentage. Other words that may indicate a relative percentage are over and under. When calculating a relative percentage, instead of dividing the final value by the reference value, divide the difference between the final value and the reference value by the reference value.

\[
\text{relative percentage} = \frac{\text{difference between final and reference values}}{\text{reference value}} \times 100
\]

Again, examine the context closely to choose the correct value for the reference value.

Question 5: In June of 2009, Philadelphia received 4.46 inches of rainfall. Normally, Philadelphia receives only 3.29 inches of rainfall in June. As a percentage, how much rainfall above normal did Philadelphia receive in June of 2009?

Answer 5: In this case, as in the last, the reference value is the normal rainfall, since this is what the 2009 rainfall is being compared against. The difference between 4.46 and 3.29 is 1.17, which makes the relative percentage \( \frac{1.17}{3.29} = 0.3556 \), or 35.6% when expressed as a percentage rounded to three significant figures.

It is not a coincidence that the answer to question five was exactly 100% less than the answer to question four. A relative percentage more than a reference value will always be 100% lower than the absolute percentage.

Question 6: Jennifer bought a television with a regular price of $750 for only $599. What percent less than the full price did she pay?

Answer 6: The difference between the regular price and the actual price is $750 – $599 = $151. Since we want to know the savings compared to the full price, the full price becomes the reference value. This gives a calculation of \( \frac{151}{750} = 0.2013 \). Rounding to two significant figures gives a 20% savings.

If we were to calculate the absolute percentage of her price for the TV, we would get \( \frac{599}{750} = 0.7987 \) or 80% when rounded. It is not a coincidence that the answer to question six and the absolute percentage added to 100%. A relative percentage less than a reference value plus the absolute percentage will always add to 100%.

When calculating the difference between the final and reference values, we always subtracted the lower number from the higher number, regardless of which was the reference value. If the final value is larger than the reference value, we will be calculating the relative percentage more than the reference value, and if the final value is smaller than the reference value, we will be calculating the relative percentage less than the reference value. In some cases, a relative percentage less than a reference value may be expressed as a negative percentage more than a reference value, especially in tables where most entries are positive. In table 2.5.1, the final column shows the percent increase in population. Since the population of Michigan declined, this is expressed as a negative increase.

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</tr>
<tr>
<td>Indiana</td>
<td>6,249,000</td>
<td>6,377,000</td>
<td>2.05%</td>
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<tr>
<td>Michigan</td>
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<td>Ohio</td>
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<td>11,486,000</td>
<td>0.31%</td>
</tr>
<tr>
<td>Wisconsin</td>
<td>5,539,000</td>
<td>5,628,000</td>
<td>1.61%</td>
</tr>
</tbody>
</table>

Table 2.5.1

Population growth in the Midwest
Question 7: Albert paid $367 for his airline ticket, while Bill, who purchased at the list minute, paid $455 for the same ticket. How much more did Bill pay than Albert? How much less did Albert pay than Bill?

Answer 7: The difference between airfares is $455 – $367 = $88. For the first part of the question, we want a relative percentage compared to Albert’s price, so we get $88 / 367 = 0.2274$, so Bill paid about 23% more than Albert. For the second part of the question, we want a relative percentage compared to Bill’s price, so we have $88 / 455 = 0.1934$, so Albert paid about 19% less than Bill.

You may be surprised that the two parts yield different numbers. This illustrates the importance of selecting the correct reference value for the calculation.

Question 8: Calculate the amount of money needed to buy a pizza for $12.99 if the price does not include the 7% sales tax.

In this problem, we need to calculate how much is 7% more than $12.99. There are two approaches that work equally well. One is to calculate the amount of the tax, and then add this to the purchase price. The other is to convert the relative percentage to an absolute percentage, and then calculate.

Answer 8 (method 1): The amount of the tax would be 7% of $12.99, which is $0.07 \times 12.99 = 0.91$. Adding this to the original $12.99 price gives $12.99 + 0.91 = 13.90$.

Answer 8 (method 2): After question five of this section, we noticed that absolute percentages are always 100% higher than relative percentages more than a reference value. Therefore adding 7% more on a price of $12.99 would be the same as calculating 107% of $12.99. This gives $1.07 \times 12.99 = 13.90$.

Question 9: A flea market offers a 15% discount on all tagged prices. Calculate the actual price for a guitar with a tagged price of $139.

Here, we must calculate how much is 15% less than $139. Again we can use either of two approaches. Either calculate the discount and subtract from the tagged price, or convert to an absolute percentage and calculate.

Answer 9 (method 1): The discount is 15% of $139, which is $0.15 \times 139 = 20.85$. Subtracting from the tagged price of $139 gives $139 – 20.85 = 118.15$.

Answer 9 (method 2): After question six of this section, we noticed that absolute percentages plus relative percentages less than a reference value always add to 100%. If the discount is 15%, the price will be 85% of the tagged price, since $100 – 15 = 85$. 85% of $139 would be $0.85 \times 139 = 118.15$.

Question 10: Carrie has $100 in her bank account. If the bank accidentally deducts 10% from her account, how much will she have left? If the bank then adds 10% of her new balance back into the account, what is her final balance?

Answer 10: This will be another illustration of the importance of reference values. The first calculation is a 10% deduction, and since 10% of $100 is $0.10 \times 100 = 10$, her balance becomes $100 – 10 = 90$. If the bank adds 10% of the new balance back, we add 10% of $90, which is $0.10 \times 90 = 9$, giving a final balance of $90 + 9 = 99$.

She still loses one dollar even though the bank restored the same relative percentage that was deducted. The amounts did not match because the reference value in the second calculation was lower.

Question 11: City council is considering raising the sales tax from 5% to 6%, arguing that a 1% increase in taxes is hardly any amount. What is the actual increase in taxes in this proposal as a percentage?
Answer 11: This problem is tricky because the reference value and the final value also happen to be percentages. The calculation is no different than if the amounts were $5 and $6 or 5 inches and 6 inches. The difference is 1%, and since we are raising the taxes from 5% to 6%, the 5% is the reference value. This gives a calculation of $\frac{1}{5} = 0.20$. City council is proposing a 20% increase in the sales tax.

This example shows how manipulating percentages can make a big increase sound small. Be aware of this trick anytime percentages are being compared. An increase of one percentage point is very different from an increase of 1%.

Exercises

1. Calculate 12% of 187.
2. Calculate 35% of 12,500.
3. Calculate 20% of the population of Texas: 27 million people.
4. Calculate six percent of the price of a $700 refrigerator.
5. Hal has finished 120 pages of his 267-page book. What percentage of the book has he read?
6. Irene has spent $21 at the movies this month. If she has budgeted $45 for the month, what percentage of her budget has she already used?
7. Jerry has completed 37 laps of a 60-lap race. What percentage of the race has he finished?
8. Kim has worked 52 hours of her 67-hour internship. What percentage of her internship has she done?
9. Lyle has donated 21 pints of blood in his lifetime, but Margaret has donated 29 pints of blood, as a percentage, how much more blood has Margaret donated than Lyle?
10. Neil took 17 minutes to complete a quiz, while Olivia took 36 minutes, as a percentage, how much more time did Olivia need than Neil?
11. Patrick bought a designer jacket for $140, while Quentin bought a similar jacket at a discount store for $59. As a percentage, how much less did Quentin spend than Patrick?
12. Roxanne used 470 minutes of her cell phone plan this month while Sheila used 380 minutes. As a percentage, how much less did Sheila talk on her phone than Roxanne?
13. Travis sent 12% more text messages this month than Upton, who sent 94 messages this month. How many messages did Travis send?
14. Veronica is buying a necklace priced at $45.00. How much will she spend after adding 8% sales tax?
15. Walter is getting a 72% discount on each issue of his favorite magazine by being a subscriber. If the cover price is $3.99, how much per issue is Walter paying for his magazine?
16. Xavier required 14% less time to run his 400-meter race than Yevgeny, who needed 58 seconds to complete the race. What was Xavier’s time in the race?
17. Zelda tips 20% at restaurants while Alex tips only 15%. As a percentage, how much more can a waitress expect from Zelda than from Alex?
18. Becky’s Money Orders used to charge a 2.9% fee for issuing money orders. She has reduced her fee to 2.5%. As a percentage, how much less is the fee now for a money order with Becky?

Is It Reasonable?

Determine whether the use of a percentage above 100% is reasonable or not in each case:

19. Median home prices in Williston, North Dakota have increased by 150% in the last five years.

20. Combining three coupons, Cliff hopes to save 120% on his next oil change.

21. 135% of the committee voted to approve the proposal.

22. After being interviewed on the news, Diane’s Twitter following increased by 140%.

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23. Carlos bought a backyard grill for $409.50, which included the 5% sales tax. What was the price before the sales tax was added.

24. Debbie and Elmo had dinner at a restaurant where they paid $53.10, which included the 18% gratuity. What was the price of the meal before the gratuity was added?
2.6 Indices

Indices (the plural of index) look a lot like percentages, where an arbitrary value is marked as the reference value. In Table 2.6.1, we have a population index for the most populous nations, where the reference value used is the population of the United States. Indices never use the per cent symbol (%).

Constructing an Index

Once a reference value is chosen, the index for any other value is calculated so that the ratio of the value to the index is the same.

In our national population index, the ratio for the United States is 
\[
\frac{100}{307,000,000} = 0.0000003257,
\]
while the ratio for Brazil comes to 
\[
\frac{62.5}{192,000,000} = 0.0000003255,
\]
and the ratio for China is 
\[
\frac{434}{1,332,000,000} = 0.0000003258.
\]
Notice that these values are the same to three significant figures, the accuracy given in the index. This leads to a formula for calculating new index values:

\[
\text{index} = \frac{\text{value}}{100} \times \frac{100}{\text{reference value}}
\]

Question 1: What would be the index for the eleventh most populous country, Mexico, with a population of 108,000,000?

Answer 1: The index for Mexico would be
\[
108,000,000 \times \frac{100}{307,000,000} = 35.2.
\]

Question 2: The population index for Germany is 26.7. Calculate the population of Germany from this index.

Now we have the index and want to solve for the value. Using more algebra:

\[
\text{value} = \frac{\text{index}}{100} \times \frac{100}{\text{reference value}}
\]

Answer 2: Multiplying the index, 26.7, by \[
\frac{307,000,000}{100}
\]
gives about 82 million.

Question 3: Construct an index for the time needed for the Olympic gold medalist to run the men’s 400-meter hurdles, based on the data in Table 2.6.2. Use the time for the 2000 Olympics as the reference value.

Answer 3: Since the reference value is the time for the year 2000, which is 47.50, the index for any year will be the time for that year multiplied by \[
\frac{100}{47.50}.
\]

1964 is 
\[
49.60 \times \frac{100}{47.50} = 104.4;
\]
1968 is 
\[
48.12 \times \frac{100}{47.50} = 101.3;
\]
1972 is 
\[
47.82 \times \frac{100}{47.50} = 100.7;
\]
and so on. The Olympic record set in 1992 would have the smallest index value, coming in at 
\[
46.78 \times \frac{100}{47.50} = 98.5.
\]
Using an Index

An index will immediately show a percentage based on the reference value. For example, the index of 62.5 for Brazil in the national population index indicates the population of Brazil is 62.5% of the population of the United States. More importantly, the ratio of two indices will indicate how much larger one quantity is than another.

**Question 4:** Using the index in table 2.6.1, determine how much larger is the population of India compared to the population of Indonesia.

**Answer 4:** The index for India is 380 and the index for Indonesia is 75.2. The ratio \( \frac{380}{75.2} = 5.05 \), tells us that India has about five times the population of Indonesia.

Of course, no index is needed to answer question four. We could accomplish the same task by taking a ratio of populations, as \( \frac{1,167,000,000}{231,000,000} = 5.05 \). In other words, the ratio of two values will always match the ratio of the corresponding indices. Since this is the case, you may wonder why indices are used.

Generally, an index is created when the number represents an average of several objects. One example is the consumer price index. The consumer price index is calculated by averaging the cost of over 200 categories of commonly purchased goods and services. Comparing the index in two time periods will reveal how the value of a dollar has changed over the period. The reference value is the average cost over the three year period from 1982 to 1984 (so no individual year’s average will correspond to 100).

**Table 2.6.3**

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</table>

**Question 5:** In 1949, Joe DiMaggio became the first baseball player to make a yearly salary of $100,000.

[2.1] Use the consumer price index to convert that salary into 2010 dollars, when Ryan Howard was given a five year contract that paid him $25 million per year. [2.2]

Since the ratio of the indices must match the ratio of the values, a little more algebra surrenders a formula:

\[
\frac{\text{second value}}{\text{first value}} = \frac{\text{second index}}{\text{first index}} \quad \Rightarrow \quad \text{second value} = \text{first value} \times \frac{\text{second index}}{\text{first index}}
\]

This looks similar to the formulas we used for unit conversion. Notice the “first” units divide out and only the “second” units remain. This guideline should help determine how to set up the conversion correctly.

**Answer 5:** To convert 1949 dollars to 2010 dollars, we multiply $100,000 by a ratio of indices. Since $100,000 is in 1949 dollars, we want our ratio to have the 1949 index in the denominator, to cancel the...
1949 units, leaving only 2010 units. $100,000 \times \frac{218.1}{23.8} = $916,387, which should be rounded to three significant figures. Joe DiMaggio's 1949 salary had the buying power of about $916,000 in 2010.

This demonstrates that, relative to inflation, the top baseball players are making 25 to 30 times as much money as they did in 1949.

**Question 6:** Compare the rates of inflation from 1973 to 1974 and from 2003 to 2004.

**Answer 6:** Inflation is the relative percentage increase of the cost of goods and services. In both of these cases, the index rose 4.9 points (49.3 – 44.4 = 4.9 and 188.9 – 184.0 = 4.9), but since inflation is a relative percentage, we will be getting very different answers. When calculating a relative percentage between two periods in time, we use the earlier time as the reference value, so the inflation rate from 1973 to 1974 would be $\frac{4.9}{44.4} = 11\%$ while the inflation rate from 2003 to 2004 would be only $\frac{4.9}{184.0} = 2.7\%$.

**Question 7:** Which gasoline is more expensive relative to consumer prices: $1.378 per gallon in 1981 or $2.392 in 2009?

This question is similar to comparing distances in miles and kilometers or weights in grams and ounces. To make the comparison, you must convert one quantity into the units of the other. In this case we could either convert $2.392$ to 1981 dollars or convert $1.378$ to 2009 dollars. It does not matter which.

**Answer 7:** First, we convert $1.378$ from 1981 dollars to 2009 dollars: $1.378 \times \frac{214.5}{90.9} = $3.252. This number is much higher than the 2009 price of $2.392, which makes the 1981 price higher when taking inflation into account.

**Exercises**

1. Construct an index based on the number of TV stations in major cities, based on table 2.6.4, using Philadelphia as the reference value.

2. Construct an index based on the number of radio stations in major cities, based on table 2.6.4, using Miami as the reference value.

For exercises 3 through 10, use table 2.6.5:

3. As a relative percentage, how much more expensive were housing prices in 2004 compared to 1998?

4. As a relative percentage, how much less expensive were housing prices in 2011 compared to 2007?

5. What should a house purchased for $94,000 in 1995 be worth in 2005, based on the housing price index?

6. In 1993, what would you have expected to pay for a house that sold for $250,000 in 2012?

7. Compare the rates of increase in housing from 1997 to 1998 and from 2006 to 2007. Which increase is higher?

8. Compare the rates of increase in housing from 1991 to 1999 and from 2001 to 2009. In which of these two seven-year periods did housing prices increase the most?
9. Which house is worth more, one bought for $80,000 in 1991, or one bought for $175,000 in 2007?

10. Which house is worth more, one bought for $250,000 in 2007, or one bought for $220,000 in 2011?

11. Based on the consumer price index, what would a ten-cent candy bar from 1955 be worth in 2010 dollars?

12. Based on the consumer price index, what would a five-dollar sandwich in 2013 have been worth in 1969?

Is It Reasonable?

13. The average prices of goods and services in the United States is guaranteed to go up every year. Is this a reasonable claim?

14. The United States housing price index should be accurate in any specific location in the United States. Is this a reasonable claim?

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15. Compare the relative increase (as a percentage) of houses from 1993 to 2013 to that of consumer goods over the same period. Which has increased more?

16. Convert the price of a house that sold for $150,000 in 2007 to its corresponding price in 1991 using the housing price index. Then convert this amount from 1991 dollars to 2007 dollars using the consumer price index. How might you interpret this calculation?
2.7 Review Exercises

1. Calculate to the correct level of precision:
   a. $\frac{346,000,000}{100,000,000}$
   b. $346 \times 1.0$
   c. $72 \times 27$
   d. $1.000 \div 7.00000$
   e. $199 \times 4$
   f. $9.7 + 0.97$
   g. $125 \div 25$
   h. $750,000 - 7500$
   i. $13.0 + 0.013$

2. Write the following numbers in scientific notation. Use the correct level of precision for calculations.
   a. $0.00000035$
   b. $346,000,000$
   c. $72 \times 10^{12}$
   d. $(2.6 \times 10^5) \times (1.7 \times 10^{-2})$
   e. $(4.92 \times 10^5) \times (3.6895 \times 10^{-7})$
   f. $\frac{2.16 \times 10^3}{3 \times 10^{-4}}$
   g. $(2.6 \times 10^5) - (2.6 \times 10^5)$
   h. $(8.92 \times 10^{-5}) + (9.92 \times 10^{-3})$
   i. $(3.00 \times 10^5) + (4.00 \times 10^{-5})$

3. Convert the following units:
   a. 360 feet to yards
   b. One day to minutes
   c. 0.5 feet to centimeters
   d. 1.00 pounds to grams
   e. Thirty ounces to kilograms
   f. 22 miles per hour to feet per second
   g. 155 Canadian Dollars to Japanese Yen
   h. 0.123 square meters to square feet
   i. 110 cubic yards to cubic millimeters
   j. 20.0 pounds per square inch to kilograms per square centimeter

4. Estimate the following to the nearest order of magnitude:
   a. If you ate one head of lettuce every hour for a year, about how many heads of lettuce would you have eaten?
   b. About how many inches are covered in the 300 mile trek from Philadelphia to Pittsburgh?

5. Calculate the following (assume exact precision):
   a. $30\% \text{ of } 500$
   b. $140\% \text{ of } 90$
   c. $20\% \text{ more than } 45$
   d. $200\% \text{ more than } 15$
   e. $8\% \text{ less than } 175$
   f. $90\% \text{ less than } 80$

6. Use table 2.7.1 to answer the following about the population of Baltimore:
   a. Since 1950, the population has (increased/decreased) by _____%.
   b. From 1900 to 2000, the population (increased/decreased) by _____%.
   c. In 1800, the population was _____% of what it was in 1850.
   d. In 1950, the population was _____% higher than it was in 1850.

7. Use table 2.7.1 to answer the following:
   a. Construct an index for the population of Baltimore, using today’s population as the reference value.
   b. Today, the population of Cincinnati is about 298,000. Assuming that the population of Cincinnati follows a similar pattern to Baltimore, use your index to predict the population of Cincinnati in 1950.
   c. In the year 2000, the population of Washington, D.C. was about 572,000. Assuming that the population of Washington follows a similar pattern to Baltimore, use your index to predict the population of Washington in 1900.

8. Use the consumer price index to convert the following:
   a. Convert $157.75, the price of an all-porcelain refrigerator in 1940, into 2013 dollars.
   b. Convert $400,000, the 2009 salary of the president of the United States, into 1960 dollars.

<table>
<thead>
<tr>
<th>Population of Baltimore</th>
</tr>
</thead>
<tbody>
<tr>
<td>1800</td>
</tr>
<tr>
<td>1850</td>
</tr>
<tr>
<td>1900</td>
</tr>
<tr>
<td>1950</td>
</tr>
<tr>
<td>2000</td>
</tr>
<tr>
<td>Today</td>
</tr>
</tbody>
</table>

Table 2.7.1
2.8 Project: Currency Conversion

1. Choose a product for which you can use the internet to check the price in a variety of locations (such as a menu item at a chain restaurant, like an Olive Garden Lasagna) or at different retail stores (such as a consumer electronics item, like a 50-inch LED television). This item will become your new currency item.

2. Determine the cost of your currency unit by looking up its price in at least six different locations or at least six different on-line retailers. If you are using different locations, try to get a mixture of regions (East, Midwest, South, and West) as well as a mixture of terrains (Urban, Suburban, and Rural). If you are using different on-line retailers, try to get a mixture of types of stores (discount, on-line only, and high-end)

3. Average these costs to get the average price of your currency unit.

4. Determine how many of the following items could be purchased with one of your new currency units. For example, how many pounds of ground beef could be purchased for one Olive Garden Lasagna, or how many grade A large eggs could be purchased for one 50-inch LED television? (Prices below from http://www.bls.gov/ro3/apmw.htm for June 2014)
   a. 1 pound of ground beef ($3.88)
   b. 1-pound head of iceberg lettuce ($1.12)
   c. Gallon of regular unleaded gasoline ($3.70)
   d. 1-pound block of natural cheddar cheese ($5.56)
   e. Grade A large egg ($0.16)
   f. 1.5-pound loaf of white bread ($2.10)
   g. 5-ounce red delicious apple ($0.44)

5. Which of these exchanges feels like the best deal? Is it really better than the other exchanges? What makes you think so, or think not?
3 Financial Applications

3.1 Budgeting

How much is that $2 lottery ticket, $4 latte, or $6 pack of cigarettes costing you over a week? Or a month? Or a year? It is good to be aware of how much money you are spending on habits like these so you can be aware of the options and alternatives should you choose to exchange a habit like this for, say, a new game system or a trip to Europe.

Most of the companies that sell these frequently-purchased, low-cost items hope the general public is not made aware of these options, as some customers might choose to stop or cut back. Some simple multiplication can be very enlightening:

Imagine purchasing a $2 lottery ticket twice a week from the national lottery, or a $4 latte four times a week from your favorite coffee shop before heading to work or school, or a $6 pack of cigarettes once per day. Looking at the chart below, the cost of these habits over a year can be significant.

<table>
<thead>
<tr>
<th>Item</th>
<th>Cost per week</th>
<th>Cost per month (4 to 4.5 weeks)</th>
<th>Cost per year (52 weeks)</th>
<th>Similar cost item to yearly cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2 lottery ticket twice a week*</td>
<td>$4</td>
<td>$16 to $18</td>
<td>$208</td>
<td>Gaming System</td>
</tr>
<tr>
<td>$4 latte four times a week</td>
<td>$16</td>
<td>$64 to $72</td>
<td>$832</td>
<td>50+ inch LED TV</td>
</tr>
<tr>
<td>$6 pack of cigarettes each day</td>
<td>$42</td>
<td>$168 to $189</td>
<td>$2184</td>
<td>One week trip to Ireland</td>
</tr>
</tbody>
</table>

*Many instant game lottery tickets pay out about 50% of the cost of the ticket in prizes, meaning these purchases average about half the cost when you consider the payouts. More about this in chapter 6.

Making these choices is not right or wrong, but simply being aware of the possibilities might be eye opening. You may find your daily latte is worth more to you over a year than a new TV. The point is to be aware of the costs and the possibilities. The next important consideration is fitting these costs into your budget. You may decide you prefer the lattes to the TV, but if your budget cannot afford either, something will need to be sacrificed.

When constructing a budget, it is important not only to consider the frequent and regular expenses like food and housing, but also the infrequent and occasional expenses like insurance payments, clothing, and textbook purchases.

Many expenses occur on a monthly cycle, such as rent or mortgage payments, car payments, utility bills, cell phone bills and cable/satellite TV bills, so calculating a budget on a monthly cycle is common. Expenses that occur more frequently (such as those in the chart above) must be multiplied out to cover a period of one month, and expenses that occur less frequently should be amortized. Even though you may not buy textbooks every month, a $300 expense three times a year can be amortized to $75 per month. Three times per year is about once every four months, and $300/4 = $75.

Question 1: Create a budget for Valerie, a college student, with the following expenses:

- Monthly share of rent payments are $450.
- Cell phone bill is $60 per month.
- Textbooks cost $250 for each of the three school sessions.
- Monthly share of utilities, including TV and internet is about $80.
- Car insurance is $500, paid twice a year.
• Weekly college meal plan is $90.
• Filling the gas tank for $40 happens about twice a month.
• Health insurance is $100 per month.
• Valerie would like to budget $75 a month for restaurants and movies.
• She would love to afford a spring break trip that would cost $550.
• Finally, she would love to keep her $4 latte on one day each week.

Spreadsheets are excellent tools for constructing budgets. The instructions presented here should work equally well with Microsoft Excel, Open Office Calc, Google Docs Spreadsheet, and many other similar programs.

**Answer 1:** Open your favorite spreadsheet program and title your columns: Category, Cost, Frequency, and Amount. Then enter the information above into the first three columns. For frequency, enter how many times a month the expense occurs. A weekly expense happens about 4 to 4.5 times a month (this example uses 4.3). An expense that happens once a year would happen $\frac{1}{12}$ times a month. A spreadsheet can calculate this by entering “$=1/12$” into a cell.

Next, in cell D2, enter “$=B2*C2$”; then grab the lower right corner of the cell and drag it down to cell D12. Finally, enter “Expenses” in cell C13 and enter “$=SUM(D2:D12)$” into cell D13. This last formula will add all the values between cells D2 and D12, giving you your total monthly expenses.

This spreadsheet lets us know that Valerie needs to bring in about $1441 a month to meet her projected expenses, whether from a part-time job, additional loans, or gifts from relatives.

Notice this budget does not include tuition, which may be covered by loans, grants, or generous parents. But more importantly, it does not include unexpected costs. Suppose a laptop needs a new battery, or you need to buy a gift for a friend’s birthday, or you want to support a local sports team by buying athletic gear. It is wise to include a miscellaneous category to cover these unexpected surprises.

**Exercises**

1. In one year, how much is spent buying a $4 magazine every week?

2. In one year, how much is spent buying a $2 app or song for a mobile device three days a week?

3. In one year, how much is spent buying a $7 fast-food lunch five days a week?

4. In one year, how much is spent buying a $1 snack from a vending machine twice a day?
In exercises 5-8, create a budget and determine the total monthly expense in each situation.

5. William has loans to cover his tuition, textbooks, and room and board on campus, his remaining expenses are:
   • $5 per week for pizza during late night study sessions.
   • $100 twice a year for his parking permit.
   • $10 per month for dues in his favorite student organization.
   • $25 per week for entertainment, usually a movie and a restaurant.
   • $8 per month for laundry tokens.
   • $50 for gas to drive home (round trip) three times during the year.
   • $40 per month for his cell-phone plan.

6. Yuna has just moved into an apartment and is figuring out her expenses:
   • $800 per month for rent
   • $100 twice a year for renter’s insurance
   • $80 per month for her public transportation pass
   • $120 per month for her cell-phone, tv, and internet bundle
   • $20 per week for her morning coffee at the coffee shop
   • $80 per week for groceries and supplies
   • $250 twice a year for clothing
   • $200 per month for entertainment and miscellaneous expenses

7. The Xylophone Corporation has the following expenses, operating from the CEO’s garage:
   • $500 every other week for the wood to construct the wooden bars.
   • $150 per month for the metal for the xylophone bases.
   • $200 four times per year for the mallets.
   • $100 per month for maintenance on woodworking machinery.
   • $1000 per week to pay for labor.

8. Zydeco Limited has the following expenses when recording their music:
   • $600 per month for renting music studio time.
   • $150 per week for hiring extra musicians
   • $100 per month for maintenance on the musical instruments
   • $200 twice a month for labels and packaging
   • $180 twice a year for software updates

Is It Reasonable?

9. Buying in bulk can save money, since many products are sold for less per unit when buying larger packages. Using this principle, is it reasonable to buy sour cream or bottled water by the gallon for a small household?

10. An internet site offered recommendations for cutting costs. One recommendation was to go grocery shopping only three times a month instead of four. Is this a reasonable approach to saving money?
3.2 Interest

Understanding how money can grow in a savings account allows you to make better choices on how and when to save money. Simple interest and periodic compounding will be covered first, but with computers now integral to every banking system, just about every savings account you will encounter will give you continuous compounding.

Simple Interest

**Question 1:** Ursula has deposited $1000 in a bank account that grants a yearly interest rate of 3%. How much money would she have in her account after one year? After five years? After 25 years?

In this example, the initial money put in the account ($1000) is called the **principal**, and is usually represented by the variable name $P$. The interest rate is often called the annual percentage rate, abbreviated $APR$.

**Answer 1:** Simple interest is a very simple formula. Each year Ursula gains three per cent of the principal she put in her account. 3% of $1000 is $0.03 \times 1000 = 30$, so every year Ursula gets another $30 in her account. This is the **interest** she earns each year. After one year, she would have $1000 + 30 = 1030$. After four years, she earns the interest four times, and would have $1000 + (4 \times 30) = 1120$. After 25 years, this increases to $1000 + (25 \times 30) = 1750$.

In general, the amount, $A$, in a simple interest earning account can be stated as $A = P + (y \times APR \times P)$, where $y$ is the number of years the money is left in the account. Using the distributive law to factor out $P$, we get the formula here:

| Simple Interest Formula: $A = P \times (1 + y \times APR)$ where $A$ is the account balance after $P$ dollars are deposited in an account for $y$ years at a rate of $APR$. |

While the formula is simple compared to what is coming, the problem with simple interest is that the money built up in the account is not earning interest. If the account is earning three per cent each year, and after one year Ursula has $1030, why shouldn’t Ursula get the interest on this $1030 during the second year, instead of using the same $1000 principal from the first year? At the start of year two, shouldn’t Ursula be allowed to earn interest not only based on her $1000 principal, but also the $30 of interest earned in the first year? The concept of earning interest based on both the principal and previously earned interest is called **compounding**.

Periodic Compounding

**Question 2:** Terrence has no desire to deposit his money in a simple interest account, and deposits his $1000 in a bank account that grants an $APR$ of 3%, compounding every year. How much money would he have in his account after each of the first four years? How much after 25 years?

**Answer 2:** The first part of this question could be answered by using the simple interest formula with $y = 1$ four times, each time adjusting the principal to include the interest earned from the previous year. After one year, his balance would match that of Ursula, since $1000 \times (1 + 1 \times 0.03) = 1030$. For the second year, the starting balance is $1030$, so the two year total would be $1030 \times (1 + 1 \times 0.03) = 1060.90$. A comparison between Ursula’s balance and Terrence’s balance for the first four years, rounded to the nearest penny, is shown here:
The difference after four years is $5.51, not a lot, but will the difference after 25 years be more significant? Calculating the interest after every year for 25 years would be time consuming, so a quicker way to calculate compounded interest would be nice. Notice that each time we calculate Terrence’s balance, we multiply the previous year’s balance by \((1 + 1 \times 0.03)\), so after two years, his balance could be calculated by $1000 \times (1 + 1 \times 0.03) \times (1 + 1 \times 0.03) = $1060.90. After three years, we get $1000 \times (1 + 1 \times 0.03) \times (1 + 1 \times 0.03) \times (1 + 1 \times 0.03) = $1092.73 and after 25 years, using this same pattern, we have $1000 \times (1 + 1 \times 0.03)^{25} = $2083.78, rounded to the nearest penny. This amount is $333.78 higher than Ursula’s balance, so while compounding may not make a much of a difference in the short term, it can help in the long term.

But why only compound the interest after each year. Why not let Terrence get his interest paid out twice in a year, so he can earn interest during the second half of the year on the interest he earned during the first half of the year. At an APR of three percent, he would earn half of that amount in half a year, or 1.5%, so after six months, Terrence would have $1000 \times (1 + \frac{1}{2} \times 0.03) = $1015. In the second half of the year, he would earn 1.5% of that $1015 balance, making his one year total $1015 \times (1 + \frac{1}{2} \times 0.03) = $1030.23. This could also be calculated by $1000 \times (1 + \frac{1}{2} \times 0.03)^2 = $1030.23. We use an exponent of two for the one year total because he had two compounding periods during the year. To find the interest after four years, we would use $1000 \times (1 + \frac{1}{2} \times 0.03)^8 = $1126.49, since there are eight compounding periods in four years. The 25 year balance would be calculated with 50 compounding periods, giving a 25 year total of $1000 \times (1 + \frac{1}{2} \times 0.03)^{50} = $2105.24.

But what is so special about twice a year?

**Question 3:** Serena finds yearly compounding too limiting and has found a savings account that grants an APR of 3%, compounding every month. How much money would she have in her account after leaving $1000 in the account for one year? After four years? After 25 years?

**Answer 3:** To find the one year balance, notice that there are 12 compounding periods, but the interest earned each month would only be one twelfth of the 3% APR. Serena’s one year total could be calculated as $1000 \times (1 + \frac{1}{12} \times 0.03)^{12} = $1030.42. In four years, there are 12 \times 4 = 48 compounding periods, so we get $1000 \times (1 + \frac{1}{12} \times 0.03)^{12\times 4} = $1127.33. And in 25 years, there are 12 \times 25 = 300 compounding periods, so we get $1000 \times (1 + \frac{1}{12} \times 0.03)^{12\times 25} = $2115.02.

The difference after four years is $5.51, not a lot, but will the difference after 25 years be more significant? Calculating the interest after every year for 25 years would be time consuming, so a quicker way to calculate compounded interest would be nice. Notice that each time we calculate Terrence’s balance, we multiply the previous year’s balance by \((1 + 1 \times 0.03)\), so after two years, his balance could be calculated by $1000 \times (1 + 1 \times 0.03) \times (1 + 1 \times 0.03) = $1060.90. After three years, we get $1000 \times (1 + 1 \times 0.03) \times (1 + 1 \times 0.03) \times (1 + 1 \times 0.03) = $1092.73 and after 25 years, using this same pattern, we have $1000 \times (1 + 1 \times 0.03)^{25} = $2083.78, rounded to the nearest penny. This amount is $333.78 higher than Ursula’s balance, so while compounding may not make a much of a difference in the short term, it can help in the long term.

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**Periodic Compound Interest Formula:**

\[
A = P \times (1 + \frac{APR}{n})^{ny}
\]

where \(A\) is the account balance after \(P\) dollars are deposited in an account for \(y\) years at a rate of \(APR\), compounded \(n\) times per year.
Continuous Compounding

Why stop at once a month for compounding? Why not once a week? Once a day? The more often the interest is compounding, the more money the savings account will make. Is there a limit to how much it can grow in a year? Yes, there is.

The following chart shows the balance in a savings account after one year with a principal of $1,000,000 with a three per cent APR at various compounding frequencies:

<table>
<thead>
<tr>
<th>frequency</th>
<th>number of periods</th>
<th>balance after one year</th>
</tr>
</thead>
<tbody>
<tr>
<td>once a year</td>
<td>1</td>
<td>$1,000,000 × (1 + 0.03/1)^1 = $1,030,000.00</td>
</tr>
<tr>
<td>twice a year</td>
<td>2</td>
<td>$1,000,000 × (1 + 0.03/2)^2 = $1,030,225.00</td>
</tr>
<tr>
<td>monthly</td>
<td>12</td>
<td>$1,000,000 × (1 + 0.03/12)^12 = $1,030,415.96</td>
</tr>
<tr>
<td>weekly</td>
<td>52</td>
<td>$1,000,000 × (1 + 0.03/52)^52 = $1,030,445.62</td>
</tr>
<tr>
<td>daily</td>
<td>365</td>
<td>$1,000,000 × (1 + 0.03/365)^365 = $1,030,453.26</td>
</tr>
<tr>
<td>hourly</td>
<td>8760</td>
<td>$1,000,000 × (1 + 0.03/8760)^8760 = $1,030,454.48</td>
</tr>
<tr>
<td>continuously</td>
<td>∞</td>
<td>$1,000,000 × e^{0.03} = $1,030,454.53</td>
</tr>
</tbody>
</table>

Table 3.2.2
Comparing different rates of compounding over one year.

There is definitely a limiting value these numbers are approaching, and the formula on the last line of the table shows how to calculate this limiting value. The number \( e \approx 2.71828 \) is a number that pops into mathematical formulas from time to time, especially in calculus applications. In this chapter, the only way you will use \( e \) is raising it to powers. Some calculators may have an \( e^x \) button, some will have an \( \exp \) button, and calculators with a text display may have an \( e^x \) button.

\[
e = \frac{1}{1} + \frac{1}{1 \times 2} + \frac{1}{1 \times 2 \times 3} + \frac{1}{1 \times 2 \times 3 \times 4} + \frac{1}{1 \times 2 \times 3 \times 4 \times 5} + \cdots
\]

\( e \) appears in many calculus applications because the rate of increase of the function \( y = e^x \) is identical to its value at every point. While there is no simple finite formula for \( e \), it is the limit of the following infinite sum:

\[
A = P \times e^{(APR \times y)}
\]

where \( A \) is the account balance after \( P \) dollars are deposited in an account for \( y \) years at a rate of \( APR \), compounded continuously.

**Question 4:** You will likely never find a bank like the ones Ursula, Terrence, and Serena found. So Reggie has no choice but to deposit his $1000 in a bank account that grants an APR of 3%, compounding continuously. How much money would he have in his account after the first year? After four years? And after 25 years?

**Answer 4:** Using the formula above, after one year he would have $1000 \times e^{0.03\times1} = $1030.45. After four years, he would have $1000 \times e^{0.03\times4} = $1127.50. And after 25 years, $1000 \times e^{0.03\times25} = $2117.00.

**Yield**

**Question 5:** Phyllis deposits $100 in a savings account that grants an APR of 5%, compounding continuously. Quinn deposits $100 in a savings account that does not grant compound interest, only giving
simple interest. What APR must Quinn’s account have for Phyllis and Quinn to have the same balance after one year?

In this example, we are trying to find the simple interest that is equivalent to 5% compounded continuously for one year. This number is called the annual percentage yield, often abbreviated APY.

**Answer 5:** First, to find Phyllis’ balance after one year: $100 \times e^{0.05 \times 1} = $105.13. This means Phyllis’ account earned $5.13 in interest. For Quinn’s account to earn $5.13 in interest, he would need an APR of 5.13% since $5.13 is 5.13% of $100. Therefore the yield on Phyllis’ account is 5.13%.

When calculating the yield of an account, it is easiest to use a starting balance of $100. This makes the interest earned equal to the yield percentage. Notice that the yield number is always slightly higher than the APR, which is why banks will usually present the yield when advertising their savings account interest rate.

**Question 6:** What is the yield on a savings account that offers an APR of 3.6%, compounded monthly?

**Answer 6:** Using a starting balance of $100 and using the periodic compounding formula for 12 periods per year, we get $100 \times (1 + \frac{1}{12} \times 0.036)^{12 \times 1} = $103.66. Subtracting the initial deposit of $100 gives a total interest earned of $3.66, which is 3.66% of $100. Therefore the yield is 3.66%.

**Exercises.**

In exercises 1-6, determine the balance in each account at the end of the time period:

1. $300 deposited into an account with 4% APR simple interest after ten years.
2. $800 deposited into an account with 3% APR simple interest after five years.
3. $700 deposited into an account with 3.5% APR compounding four times a year (called quarterly) after three years.
4. $450 deposited into an account with 5% APR compounding monthly after eight years.
5. $1000 deposited into an account with 4.2% APR compounding continuously after 15 years.
6. $1500 deposited into an account with 3.8% APR compounding continuously after 20 years.
7. What is the annual yield (to three decimal places) for the account described in exercise 5?
8. What is the annual yield (to three decimal places) for the account described in exercise 6?

**Is It Reasonable?**

9. Is it reasonable for an account to offer an interest rate of 4% APR but an annual yield of 3.9%?
10. Is it reasonable for an account to offer an interest rate of 2% APR but an annual yield of 10%?

**Advanced Applications**

11. How long will it take to double the initial investment in an account offering 5% APR compounding continuously?
12. How long will it take to triple the initial investment in an account offering 4% APR compounding continuously?
3.3 Savings Plans and Loan Payments

Savings Plans

**Question 1:** Olivia is setting aside money to buy a grand piano. Each month, she deposits $100 into an account offering 4% APR, compounding monthly. How much money will she have after one year?

**Answer 1:** One way to picture this problem is to imagine $100 being deposited into a different account each month. The first $100 she deposits gets the benefit of 12 compounding periods, the next $100 she deposits gets the benefit of only 11 compounding periods, and the next gets 10 periods, and so on. After calculating these 12 balances on these 12 separate accounts, the balances can be added for a final total.

If you think 12 calculations would get tiring, imagine calculating the balance after five years (60 calculations), or 20 years (240 calculations). Fortunately, there is a formula that can be used to calculate this balance.

**Monthly Savings Plan Formulas:**

\[ (1) \ A = P \times \frac{\left( 1 + \frac{APR}{12} \right)^{12y} - 1}{\frac{APR}{12}} \]

\[ (2) \ P = A \times \frac{\left( \frac{APR}{12} \right)^{12y} - 1}{\left( 1 + \frac{APR}{12} \right)^{12y} - 1} \]

where \( A \) is the account balance after \( P \) dollars are deposited each month into an account for \( y \) years at a rate of \( APR \), compounded monthly.

**Answer 1 (again):** Armed with the first formula, we can calculate: $100 \times \frac{\left( 1 + \frac{0.04}{12} \right)^{12 	imes 1} - 1}{\frac{0.04}{12}} = $1222.25

Solving this formula for \( P \), we get the second formula, which determines how much needs to be saved each month to reach a certain target at a certain time.

**Question 2:** Nick is saving for a $2500 down payment on a car. If he wants to have $2500 in two years, how much must be deposit each month into an account that offers an APR of 3%?

**Answer 2:** Using the second formula above, we can calculate $2500 \times \frac{\left( 1 + \frac{0.03}{12} \right)^{12 	imes 2} - 1}{\left( 1 + \frac{0.03}{12} \right)^{12 	imes 2}} = $101.20.

The power of savings plans is the long term benefits. Starting a retirement account early can make a big difference in retirement savings.

**Calculator Tips**

If you are using a calculator with a text based display, like a TI-83, you can enter the formula as it appears, just be sure to use parentheses around the entire numerator and denominator when fractions have complex expressions, so \( \frac{9 + 3}{8 - 5} \) would be entered as \( (9 + 3) / (8 - 5) \).

If you are using a scientific calculator with only a numeric display, you may have to save some of your work by writing partial results on paper or using your calculator’s memory. In calculating answer 2, first note the value of the exponent, 12\( \times 2 = 24 \). Then calculate the value of the denominator by entering \( 0.03 [ \div ] 12 [ \times ] 1 [ \times ] 24 [ - ] 1 \). Then store this in memory or use the \( [ V ] \) key to make this value a denominator. Then finish the calculation.
Question 3: Melissa starts saving $100 a month in a retirement account earning an APR of 5% at the age of 25. How much money will she have in the account after 40 years? Compare this to Leon, who starts saving $100 a month in the same kind of account at age 40. How much will he have after 25 years?

Answer 3: Calculating Melissa’s account balance, we get $100 \times \left(1 + \frac{0.05}{12}\right)^{12 \times 40} - 1 = $152,602.02. For Leon, we get $100 \times \left(1 + \frac{0.05}{12}\right)^{12 \times 25} - 1 = $59,550.97. Even though Melissa was only saving for a period 60% longer: \((40-25)/25 = 0.60\), she saved more than 2.5 times what Leon saved, an advantage of about 156% over Leon’s total savings. \((152,602-59,551)/59,550.97 \approx 1.56\)

Loan Payments

If you cannot wait to save up for a large expense, a bank will lend you money, which can be paid off in monthly installments. The formula looks very similar to the savings plan, except that it works in reverse.

Monthly Loan Payment Formulas:

\[
(1) \quad P = L \times \frac{\left(\frac{APR}{12}\right)}{1 - \left(1 + \frac{APR}{12}\right)^{-12y}}
\]

\[
(2) \quad L = P \times \frac{1 - \left(1 + \frac{APR}{12}\right)^{-12y}}{\left(\frac{APR}{12}\right)}
\]

where \(P\) is the monthly payment for a loan of \(L\) dollars over a term of \(y\) years at an interest rate of \(APR\).

Question 4: Katya needs a loan for a $30,000 addition to her home. The bank is offering a loan at 6% APR, repaid over 15 years. What will her monthly payment be?

Answer 4: Using the first loan payment formula, we see that her payment, rounded to the nearest penny, would be $30000 \times \frac{0.06}{12} \times \frac{1}{1 - \left(1 + \frac{0.06}{12}\right)^{-180}} = $253.16.

Banks will handle rounding by changing your payment by a penny on occasion. The more precise answer to question 4 is $253.15705, so most months the payment due would be $253.16, but every three or four months, the payment would be reduced to $253.15.

Question 5: James can afford a car payment of $300 per month. If the bank is offering a loan at 5% APR, repaid over five years, how much can he afford to borrow?

Answer 5: Using the second loan payment formula, we see that his $300 a month over five years can support a loan amount of $300 \times \frac{1 - \left(1 + \frac{0.05}{12}\right)^{-12\times5}}{\left(\frac{0.05}{12}\right)} = $15,897.21.

Knowing how big a loan you can afford before shopping for a big item can help when planning these purchases. Knowing that James can afford an auto loan of about $15,900, lets him know that he would have to save up for a $2100 down payment on a car that costs $18,000.

Question 6: After Katya makes her first payment on her $30,000 loan, what is Katya’s new loan balance? What is her balance after two loan payments?
Tracking this information, how much of each payment is interest and how much of each payment lowers the principal, is called an amortization schedule.

Answer 6: An annual loan at 6% APR would be $6%/12 = 0.5\%$ each month. $30,000 \times 0.005 = $150, so $150 of the first $253.16 payment pays the interest, and the remaining $103.16 reduces the $30,000 principal down to $29,896.84. With the second payment of $253.16, we need $29,896.84 \times 0.005 = $149.48 to pay the interest and the remaining $103.68 reduces the principal to near $29,793.16.

The table at the right shows the amortization schedule for the first and last six payments of the loan. Notice that during the first year, more of the payments are applied to interest than to principal, while at the end of the loan, very little interest remains and most of the payment is lowering the principal. Due to rounding, sometimes the total payment adds to $253.15.

The longer the term of the loan, the higher the contrast between interest and principal amounts at the beginning and end of the loan. The next section will show a graph of this relationship over a 30 year mortgage term.

<table>
<thead>
<tr>
<th>Payment</th>
<th>Interest</th>
<th>Principal</th>
<th>Balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
<td>$150.00</td>
<td>$103.16</td>
<td>$29897</td>
</tr>
<tr>
<td>2nd</td>
<td>$149.48</td>
<td>$103.67</td>
<td>$29793</td>
</tr>
<tr>
<td>3rd</td>
<td>$148.97</td>
<td>$104.19</td>
<td>$29689</td>
</tr>
<tr>
<td>4th</td>
<td>$148.44</td>
<td>$104.71</td>
<td>$29584</td>
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<tr>
<td>5th</td>
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</tr>
<tr>
<td>180th</td>
<td>$1.26</td>
<td>$251.90</td>
<td>$0</td>
</tr>
</tbody>
</table>

Table 3.4.1 Amortization schedule for a loan.

Is it right to plan financially?

This famous quote from the Sermon on the Mount could be interpreted to mean it is wrong to plan for one’s future:

Therefore I say to you, do not worry about your life, what you will eat or what you will drink; nor about your body, what you will put on. Is not life more than food and the body more than clothing? Look at the birds of the air, for they neither sow nor reap nor gather into barns; yet your heavenly Father feeds them. Are you not of more value than they?

Therefore do not worry, saying, ‘What shall we eat?’ or ‘What shall we drink?’ or ‘What shall we wear?’ For after all these things the Gentiles seek. For your heavenly Father knows that you need all these things. But seek first the kingdom of God and His righteousness, and all these things shall be added to you. Therefore do not worry about tomorrow, for tomorrow will worry about its own things. Sufficient for the day is its own trouble.

Matthew 6:25,26,31-34

However, there are plenty of instances where planning for the future is encouraged, such as Joseph advising the Egyptians to set aside stockpiles of grain during seven years of plentiful harvests, which would be followed by seven years of lean harvests. Emanuel Swedenborg also states in his theological writings that it is a law of divine providence that people should act in freedom according to reason. People are given the ability to reason and determine the likely outcomes from their actions, a gift that should be used to plan for the future. [3.1] The message to be taken from this passage in Matthew is that people should not spend energy worrying about such plans once they have been decided and implemented, instead trusting that God’s Divine Providence is leading us toward a happy state in the long term. Whether our plans work or not, we should continue to use our reason to act in the wisest fashion given the current situation. Expect some plans to work as predicted and others to need several revisions. For those that trust in the divine, Swedenborg states:
“Unruffled is their spirit whether they obtain the objects of their desire, or not; and they do not grieve over the loss of them, being content with their lot. If they become rich, they do not set their hearts on riches; if they are raised to honors, they do not regard themselves as more worthy than others; if they become poor, they are not made sad; if their circumstances are mean, they are not dejected. They know that for those who trust in the Divine all things advance toward a happy state to eternity, and that whatever befalls them in time is still conducive thereto.”

_Arcana Coelestia_, no. 8478

Exercises.

1. How much money will Alexis have after depositing $100 every month for 20 years into an account offering 4% APR compounding monthly?

2. How much money will Bill have after depositing $20 every month for five years into an account offering 3.2% APR compounding monthly?

3. How much should Chantal deposit into an account each month for two years, at 3.5% APR compounding monthly, in order to have saved enough for a $500 down payment on a car?

4. How much should Don deposit into an account each month for five years, at 3.7% APR compounding monthly, in order to have saved enough for a $5000 down payment on a house?

5. Calculate Edith’s monthly mortgage payment on a loan of $200,000, paid back over 30 years, with an APR of 6%.

6. Calculate Floyd’s monthly payment on a loan of $3000, paid back over five years, with an APR of 5%.

7. How large a loan can Gina get with a 4% APR loan, if she can afford a $250 payment every month for five years?

8. How large a loan can Hiram get with a 5.5% APR loan, if he can afford a $1300 payment every month for 30 years?

Is It Reasonable?

9. Is it reasonable for a bank to offer a higher APR on a savings account than on a loan? Why or why not?

10. Is it reasonable to find a bank that is offering a loan APR that is lower than the current rate of inflation? Why or why not?

Advanced Applications

11. If payments are made at a different frequency, replace the number 12 in the formulas by the frequency of the payment. Calculate Justin’s mortgage payment on a $250,000 loan over 30 years with 5% APR, if he makes a payment every two weeks (26 times per year).

12. If the interest is not compounded monthly, the expression \((\text{APR}/12)\) can be replaced by the monthly yield. How much money will Irene have saved by depositing $150 every month for five years, into an account offering 4% APR compounded continuously.
3.4 Loan Structures

Mortgages

Mortgages are usually the largest loans an individual will ever carry (Businesses and governments will often deal with much larger loans). Many first mortgages are given a 30 year term, and as you saw with Katya’s home improvement loan amortization schedule, more of the initial payments of the loan get applied to interest while less is applied to the principal. The first graph shows how much of each of the 360 payments in a 30 year mortgage on a $240,000 loan is interest, and how much is principal. The second graph shows how much of the $240,000 has been paid off according to the same loan arrangement. Using the formula from the last section, a $240,000 loan at 6% interest requires a monthly payment of about $1438.92.

Looking at the first chart, about $1200 of the first $1439 payment is interest, so only about $239 is used to lower the principal. It is not until 18.5 years have passed that the principal portion of a payment will finally be larger than the interest portion. As a result, the loan is not being paid at a linear rate, as seen in the second graph. It takes 21 of the 30 years for half of the loan to be paid off, while the other half is being paid in the final 9 years. One final interesting fact about this loan is that making a $1438.92 payment 360 times is a grand total of about $518,000, more than double the amount of the loan.

Most mortgages offer no penalty for paying the loan early. A few extra dollars, especially early in the loan period, can mean paying the mortgage sooner, saving thousands at the end of the mortgage. Paying an extra $60 per month, about $2 per day, beyond the $1438.92 payment means that an extra $60 of principal is paid every month, thus lowering the amount of the interest portion of the next payment. Over the life of the 30 year loan in this example, the loan is paid off three years earlier. The extra $60 paid 324 times ($19,440) saves 36 payments of $1438.92, almost $52,000 of payments.

Some mortgages offer a lower interest rate in exchange for extra cash up front. This extra cash is put into units called points, where one point is equal to 1% of the borrowed amount.

Question 1: Idris has been offered a $240,000 loan at 6% APR, but the bank has offered to lower the interest rate to 5.5% if she pays two points. Is she better off using the two points to lower the starting principal, keeping the 6% rate, or is she better off using the two points to lower the interest rate?
Answer 1: Two points would be 2\% of the $240,000 loan amount, or $4800. Using this money to lower the principal lowers the loan amount by $4800 to $235,200. The payment using the formula from the last section becomes $1410.14. On the other hand, if the points are used to lower the interest rate, the principal starts at $240,000, and the 5.5\% rate gives a monthly payment of $1362.69, a savings of almost $50 a month.

Why would a bank offer the same mortgage with two different monthly payments? Purchasing points to lower an interest rate does not pay down any principal. Therefore, if the mortgage is refinanced because Idris can access a lower interest rate, either from interest rates dropping or having her credit rating improve, the money used to purchase the lower interest rate is lost.

Auto Loans

Similar to mortgages, auto loans have the same type of curve, where initial payments mostly pay interest and later payments mostly pay principal. However, instead of offering points, many dealerships offer the customer a choice between “cash back” or a very low interest rate (but not both). The car dealerships advertise the interest rate and the “cash back” amount, but rarely advertise the price of the vehicle, which is artificially raised to allow the dealership a chance to make a healthy profit, even after the “cash back” or low interest rate is used.

Question 2: Harvey is buying a car for $26,000. The dealership offers a “cash back” incentive of $1500 or an interest rate of just 3\% APR over a five year loan. Without the special interest rate, Harvey can get a car loan with an APR of 5\% over a five year loan. Is Harvey better off financing the $26,000 at 3\% APR, or should he use the $1500 cash to pay down the principal, financing the remaining $24,500 at 5\% APR?

Answer 2: Using the loan formulas once again, $26,000 at 3\% APR over five years gives a monthly payment of $467.19. $24,500 at 5\% APR over five years gives a monthly payment of $462.35, so Harvey is slightly better off using the cash to pay down the loan principal rather than using the lower interest rate.

Of course, if Harvey has a lot of higher interest debt, like credit card debt, he might be best off financing the whole $26,000 at the 5\% APR and then using the cash to pay down a higher interest credit card balance.

Credit Cards

Credit Cards offer the most flexible loan terms of any structured loan. Once you have a line of credit with a card company, you may borrow any amount up to your credit limit, and pay back a small amount, a large amount or the entire amount each month. Credit card debt is unsecured, since you are not surrendering any assets if you find yourself unable to pay back the credit card debt, unlike auto loans or mortgages, where failure to pay over a period of time leads to repossession of the car or foreclosure on a home. If a customer fails to repay at least the minimum amount due on a credit card debt, the card company cannot force the customer to pay, but they can report the non-payment to credit bureaus which track the payment history of customers. If someone has a history of not paying credit card debt, it will be very difficult to get other types of secured loans, such as mortgages and auto loans.

Since credit card debt is unsecured, the interest rates are generally much, much higher. Introductory rates can start around 10\%, but many cards have rates between 20\% and 25\%. Using credit cards for long term loans can make credit card companies very rich, yet thousands of people have credit card debt in the thousands of dollars because it is so easy to buy that “one more item,” especially if it is “on sale!”

Question 3: Gina has a credit card with a balance of $2000 and an interest rate of 23\%. About how long will it take Gina to pay off this credit card balance making a payment of $40 each month?

Answer 3: While we do not have a loan payment formula solved for the variable $y$, the number of years. We can use the second loan payment formula to calculate the amount that can be borrowed using a payment of $40 over any number of years. We can plug in $y=3, y=4, y=5$, and so on, until the loan amount
reaches or exceeds $2000. With y=3, we see that $40 a month at 23\% \text{ APR} pays a debt of about $1033. Increasing to six years, the debt covered only increases to about $1555. We have to get to 14 years before the debt amount reaches $2000. This means that Gina will have to make 14 years of monthly payments of $40, that’s $40 \times 12 \times 14 = $6720, just to pay off the $2000 balance.

The reason it can take so long to pay off the loan is that the monthly interest on $2000 at 23\% is quite high, $2000 \times \left(\frac{0.23}{12}\right) = $38.33, meaning that only $1.67 is going to the principal on the first payment. Making only minimum payments on credit cards for an extended period of time is not financially wise. If you use a credit card, try to pay most if not all of the balance each month.

**The Housing “Bubble”**

In the years leading up to 2008, housing prices were increasing at their fastest rate over the last 20 years (see the housing index in problem 2.6.5). Then housing prices suddenly dropped. In some regions, prices fell by 50\% or more in a short period of time. This created a crisis that threatened banks, slowed the economy, and gutted retirement accounts. What exactly happened?

The equity in a home is the value of the home minus the loan principal owed to the bank. When housing prices increase, the terms of the loan do not change, so all of the increase in value goes to the home owner in terms of increased equity. If the home is then sold, this equity becomes profit for the home owner. Alternatively, the home owner could seek a new loan, borrowing a larger amount (since the home is worth more), and using the cash to take a luxurious vacation or invest in more homes with the hope these homes also increase in value.

Since down payments on homes were not required to be very large, a small investment of a few thousand dollars down payment could give the home owner a mortgage on a home of several hundred thousand dollars. If that home’s value were to increase by $50,000 dollars in one year, that home owner could sell the home, or refinance to get access to that $50,000 of equity, which could become the down payment for five more homes, and the cycle continues. As more homes were purchased, demand for houses would increase, increasing prices further. Real estate in Florida and Arizona, states with quickly increasing populations, became very appealing for investors.

Banks had no problem with this arrangement, since every mortgage comes with banking fees attached. Every mortgage a bank sells would be worth a few points in closing costs. Banks became so eager to loan, that they were offering mortgages to owners who had no chance of affording the payments for more than a year. But this would not matter since the home’s value would increase, and the homeowner could simply resell or refinance after a year, and reap a large profit. Banks were making money and homeowners were making money. No one was losing.

Until housing prices stopped increasing.

Just as the home owner gets all of the increase of a home’s value in terms of increased equity, so the home owner suffers all of the decrease when the home loses value, in terms of lost equity. A $300,000 home with a $240,000 mortgage gives its owner $60,000 of equity. If the home value increases by $50,000, the equity increases to $110,000. If the home value decreases by $50,000, the equity decreases to $100,000. And if the home value decreases by $100,000, the equity becomes negative, at minus $40,000. And this is where all the problems happen.

With small down payments, it only takes a small decrease in home values to make the equity negative. When a home owner owes a loan principal higher than the value of the home, the mortgage is called underwater. If the homeowner continues to make payments, the banks still get their money, and there is no problem, other than the homeowner who has lost some equity on the home. But many homeowners at this time were not planning on continuing payments. Their plan was to sell after a year to make a tidy profit, but as mortgages went underwater, homes could not be resold, and homeowners were stuck with a house that could not be sold. These homes would go into foreclosure.
When a home is foreclosed, the bank offering the mortgage takes ownership of the home and attempts to resell it to get the money back that was loaned to the homeowner. But if the mortgage is underwater, the bank cannot recover the full amount of the mortgage, and the bank takes a loss. Even worse, when the banks try to resell these many homes, the demand for homes in these areas decreases further, lowering prices even more. If the bank loses enough money on these loans, the bank fails to have the assets needed for daily operations, such as covering withdrawals from its account holders. If the account holders cannot withdraw money to pay bills and buy products (or even if the account holders think this might be a possibility), account holders buy less. When less is bought, companies don’t make sales quotas, making the value of the companies (and their stock) decrease. Companies try to compensate by reducing their workforce, so more people now have less money to buy products, and the cycle continues.

Exercises.

1. Katie’s mortgage company is offering to lower her interest rate from 5.5% to 5.4% if she will pay one point on her 30-year, $180,000 home loan. By how much (if at all), will this lower her monthly payment, compared to using the one point to lower her loan principal with the higher interest rate?

2. Leonard’s mortgage company will lower his interest rate from 4.9% to 4.5% if he will pay two points on his 15-year, $200,000 home loan. By how much (if at all), will this lower his monthly payment, compared to using the two points to lower his loan principal with the higher interest rate?

3. Marge is financing $16,000 of the cost of her new car. The dealer is offering an interest rate of 1% over four years, or $1000 cash. Calculate the monthly payments two ways: (1) financing $16,000 at 1% over four years, and (2) using the cash to lower the principal to $15,000, and financing at 5% over four years.

4. Ned is financing only $8,000 of the cost of his new car, since he had saved up a large down payment. The dealer is still offering an interest rate of 1% over four years, or $1000 cash. Calculate the monthly payment two ways: (1) financing $8,000 at 1% over four years, and (2) using the cash to lower the principal to $7,000, and financing at 5% over four years.

Is It Reasonable?

5. Is it reasonable to expect that doubling the interest rate on a loan will mean that the amount of interest paid over the life of the loan will also double?

6. If a company buys a home for the purpose of remodeling the home over a few months, and then reselling the home for a much higher price, would it make sense for the company to purchase a lower interest rate?
3.5 Taxes

Taxes are collected by governments to allow governments to provide services to the public. There are many kinds of taxes: sales taxes, property taxes, and income taxes are the most common types, but governments can also raise money by imposing fares or tolls on certain modes of transportation. Tax rates are set by these governments and payment of them is enforced by the government.

Types of Tax

A **lump-sum tax** is a tax where every person pays the same amount, regardless of income, status, or wealth. The most common type of lump-sum tax is a **poll tax**, frequently used from ancient times up until the 19th century. An original definition of “poll” was “head”, meaning this tax was to be applied to each person, regardless of any intent to vote. In later times, poll taxes were enforced only on minorities whom a government wanted to suppress during elections, and thus the perceived meaning of a poll tax was changed.

Other examples of lump-sum taxes are tolls. Each car that crosses a toll bridge pays the same lump sum amount, regardless of the income, status, or wealth of the driver. Lump-sum taxes are not used much today since they are regressive.

A **regressive tax** is a tax where individuals with a lower income pay a higher proportion of their income as tax when compared to individuals with a higher income. Lump-sum taxes are regressive since any lump sum amount would represent a higher proportion of income to a lower income individual. Suppose a state imposed a $20 tax when applying for state identification, such as a driver’s license. To a person with an income of $20,000, this tax would be 0.1% of income, but to a person with an income of $1,000,000, this tax would only be 0.002% of income.

Public transportation fares can be considered a regressive tax since those with lower incomes are more likely to use public transportation on a regular basis. For this reason, most cities will not raise fares on public transportation to pay for other city services, such as education or law enforcement.

While lotteries are often regressive, since lower income individuals are more likely to purchase these tickets compared to higher income individuals [3.2], lotteries cannot be considered taxes since they are a voluntary purchase, rather than a mandatory fee attached to a purchase, like a sales tax. There will be more discussion on the wisdom, or lack thereof, on lotteries in the chapter on probability.

A **flat tax** is a tax where individuals pay the same percentage of tax, usually based on income. Most local and some state governments collect income tax as a flat tax. For example, Pennsylvania has a flat income tax rate of 3.07%, while Colorado collects an income tax of 4.63%. [3.3]

Property taxes are a type of flat tax, based on the value of a home and lot instead of income, and are paid by the property owner. Most property tax rates are between 0.2% and 2% of the assessed value of the property. [3.4] Renters would not pay this tax, though most landlords would account for this cost when setting the rental rate.

Sales taxes are another type of flat tax, though these taxes are based on the amount of a purchase. Sales taxes are usually a flat rate of a purchase. In Oklahoma, for example, the state sales tax is 4.5%, though individual cities can add their own tax, increasing this number to as much as 11%. [3.5] Sales taxes can be regressive, since those with a lower income are likely to spend a larger proportion of their income, while those with higher income are more able to put some income into savings plans or investments. Some states, like Pennsylvania, reduce this effect by not imposing sales taxes on essential items, such as food, clothing, or personal hygiene products. [3.6] Other states, like Oregon, almost avoid sales taxes completely, opting instead for a graduated income tax that is progressive. [3.3, 3.5]
Graduated Income Tax

A progressive tax is a tax where individuals with a higher income pay a larger proportion of that income in taxes. The United States federal government and many state governments use a graduated income tax where different marginal tax rates are used in different income ranges.

Table 3.5.1 shows the tax brackets and rates for the year 2014 in the United States. A married couple has two choices. Each spouse can file separately as a single filer, or they can combine their income and file jointly. Unmarried filers who have dependents have the option to file as a head of household. The numbers in this chart are updated every year using the change in the consumer price index, which was studied in chapter 2.

One important feature of a marginal tax structure is tax is paid on income in each category according to its rate, regardless of the total income. A single filer who earns $50,000 per year, will not simply pay a 25% tax rate. He or she will pay 10% tax on the first $9,075, then 15% on the next $27,825 (which is 36,900 – 9,075), and then 25% on the final $13,100 (which is 50,000 – 36,900). This system avoids what would be a stiff penalty for just barely crossing a boundary into a higher tax bracket.

If the top rate were applied to all income, a single filer would pay 25% on an income of $37,000, resulting in $9,250 tax and $27,750 of income left, while a single filer would pay 15% on an income of $36,000, resulting in $5,400 tax and $30,600 of income left. This would be an almost $3,000 penalty on earning that additional $1000.

**Question 1:** Cathy has an adjusted gross income of $50,000 in 2014. Calculate how much tax she owes if she is a single filer.

**Answer 1:** As mentioned in the paragraph above, Cathy needs to calculate different tax amounts on each bracket until she reaches $50,000. She owes 10% on the first $9,075; 15% on the next $27,825, bringing her through the second bracket to $36,900; and then 25% on the final $13,100, bringing her total to $50,000. Adding up these amounts gives a total of $8,356.25 owed in taxes.

<table>
<thead>
<tr>
<th>Bracket</th>
<th>Amount</th>
<th>Rate</th>
<th>Tax</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 to 9,075</td>
<td>9,075</td>
<td>10%</td>
<td>$907.50</td>
</tr>
<tr>
<td>9,075 to 36,900</td>
<td>27,825</td>
<td>15%</td>
<td>$4,173.75</td>
</tr>
<tr>
<td>36,900 to 50,000</td>
<td>13,100</td>
<td>25%</td>
<td>$3,275.00</td>
</tr>
<tr>
<td><strong>Total tax owed:</strong></td>
<td></td>
<td></td>
<td><strong>$8,356.25</strong></td>
</tr>
</tbody>
</table>

**Question 2:** Daniel and Edith have a combined adjusted gross income of $250,000 in 2014. Calculated how much tax they owe if they file jointly as a married couple.

**Answer 2:** At $250,000, they will work through five different brackets to calculate their tax. The calculations at each of the five brackets is shown to the right. Multiplying the amount in each bracket by the rate of the bracket gives the amount of tax in each bracket, and adding these numbers gives a total tax owed of $58,404.50.

You may notice the phrase adjusted gross income was used in each of these questions. This is because the total gross income can be reduced through deductions before calculating the tax owed. Common deductions include charitable contributions, some types of medical expenses, interest payments on
mortgages, qualifying tuition costs, and exemptions for each dependent person in the household. After the tax owed is calculated, **tax credits** are applied to the amount, which can reduce the amount of tax owed. These are less common than deductions, however, **taxes withheld** from a paycheck will act like a tax credit when it comes time to file. If the sum of your tax credits and taxes withheld is more than the tax you owe, the government will issue a **tax refund**. While this refund can feel like winning a small lottery, what this really means is that you probably withheld more taxes than necessary, and you just gave the government an interest free loan to hold your money for a year.

**Exercises.**

In Exercises 1 to 4, identify whether the “tax” described is a lump-sum, flat, or progressive tax.

1. An office collects $5 from each person to pay for a gift for their supervisor.

2. A cosmetics company takes 15% of the profit from each representative to pay for administrative costs and advertising.

3. A church organization requests that each member in the congregation donates 5% of their income.

4. Condominium fees at residence building are determined by the assessed value of the condominium. Those worth about $200,000 have an annual fee of $100, those worth $300,000 have an annual fee of $200, and those worth $400,000 have an annual fee of $300.

5. Calculate how much tax Felton owes if he files as a single person and makes an adjusted gross income of $150,000 in 2014.

6. Calculate how much tax Greta owes if she files as a head of household and makes an adjusted gross income of $150,000 in 2014.

7. Calculate how much tax Henri owes if he files as a head of household and makes an adjusted gross income of $65,000 in 2014.

8. Calculate how much tax Isabelle and Isaac owe if they file jointly and make an adjusted gross income of $65,000 in 2014.

**Is It Reasonable?**

9. Is it reasonable to assume that Jacob’s taxes will be similar (assume an income around $40,000) with a $4000 tax deduction or a $4000 tax credit?

10. Before applying tax credits and taxes withheld, is it reasonable to assume that if Kendra has an adjusted gross income higher than Lionel, then Kendra’s federal taxes will be higher than Lionel’s?

**Advanced Applications**

11. Design a spreadsheet to calculate the tax owed for someone filing as single, according to table 3.5.1.

12. Modify the spreadsheet from exercise 11 to show the tax owed when filing in each of the three possible categories.
3.6 Review Exercises

1. Create a monthly budget for the expenses for Fred’s Chocolate based on this list:
   - Monthly rent for the factory is $2400
   - Sugar is delivered weekly, costing $500 for each delivery
   - Milk products are delivered daily, costing $100 for each delivery
   - Truffle powder is delivered four times a year, costing $12000 for each delivery
   - Coffee beans are delivered every other week, costing $250 for each delivery
   - Citrus fruits are delivered twice a week, costing $300 for each delivery

2. Calculate the total spent in one year to support the following habits:
   a. Smoking two packs of cigarettes a day, when a pack costs $7.
   b. Drinking a $4 latte three times a week.
   c. Spending $15 on a movie (including popcorn and soda) twice every weekend.
   d. Spending $50 a month on high speed internet access.

3. Calculate the value of the following investments.
   a. $100 invested for two years at 9% APR simple interest.
   b. $200 invested for one year at 8% APR compounded monthly.
   c. $300 invested for six months at 7% APR compounded daily.
   d. $400 invested for five years at 6% APR compounded continuously.
   e. How big must the initial investment be to get $1000 at the end of one year if the interest is 5% APR compounded monthly.

4. Calculate the value of the following savings plan:
   a. Depositing $300 every month for a year at 8% APR compounded monthly.
   b. Depositing $150 every month for two years at 8% APR compounded monthly.
   c. How much must be deposited each month in a savings plan to have $10,000 dollars at the end of five years. (use 8% APR compounded monthly)

5. Calculate the APY (yield) for each of these accounts.
   a. 9% APR compounded annually.
   b. 8% APR compounded monthly.
   c. 7% APR compounded daily.
   d. 6% APR compounded continuously.

6. Calculate the monthly loan payment for the following loans.
   a. Financing $15,000 over five years at 7% APR.
   b. Financing $250,000 over 30 years at 5% APR.
   c. Financing $1000 over one year at 20% APR.

7. Calculate the federal tax owed based on table 3.5.1.
   a. Adjusted gross income of $95,000, filing as single.
   b. Adjusted gross income of $550,000, filing jointly as a married couple.
   c. Adjusted gross income of $51,000, filing as a head of household.

8. Describe the difference between the following terms.
   a. Simple interest and compound interest.
   b. Regressive tax and progressive tax.
   c. Lump-sum tax and flat tax.
   d. Tax deduction and tax credit.
3.7 Project: Lottery Lump-Sum Payouts

When a lottery prize is announced, the present day value of the winnings is a lot less than the announced prize. The announced prize is the total of all payments. It is similar to the mortgage example in section 3.4, where over a $240,000 mortgage at 6% over a 30-year term, the homeowner makes about $518,000 of payments over the life of the loan. If these were lottery payments, the prize would be announced as $518,000 even though the value of the payments in today’s dollars is only $240,000.

In chapter 6, we will cover how to calculate the probability of winning one of these grand prizes, as well as the expected value from playing such a game (it is rarely in your favor).

You can calculate the present-day value of a lottery prize if you know (or estimate) the interest rate that the lottery company can access by using the reverse loan payment formula. In this example, we will use a 1.5 million dollar prize paid out monthly over a ten year term.

1. Calculate the amount of each payment.
   Ten years would prove 120 monthly payment. $1,500,000/120 = $12,500 per month.

2. Determine (or estimate) the interest rate.
   Assume that the loan company has access to a savings account with a 3% APR for this example.

3. Use the second loan payment formula from section 3.3 to determine what loan amount could be supported by this payment.
   Over ten years, a loan payment of $12,500 per month would support a loan amount of:
   \[
   $12,500 \times \frac{1 - \left(1 + \frac{0.03}{12}\right)^{-12\times10}}{\left(\frac{0.03}{12}\right)} = $1,294,522, \text{ when rounded to the nearest dollar.}
   \]

If you have a lottery prize that is paid out weekly, replace each 12 in the formula by 52.

Calculate the present day value of the following lottery prizes:
1. A $10 million prize, paid out over 15 years, where the lottery has access to 3.5% APR.
2. A $150 million prize, paid out over 25 years, where the lottery has access to 5% APR.
3. A $1000 a week prize, paid out over 30 years, where the lottery has access to 4% APR.
4. Look up the current U.S. Powerball jackpot size and payout terms (how many years). Then visit a major bank’s website to see what interest rate is offered on a CD with a similar length term.

Then answer the following questions:
5. Does the relative value, compared to the announced prize, go up or down when the payout period is lengthened?
6. Why would a lottery prefer to advertise the total amount of the payouts, rather than the present-day value of the prize?
7. [Advanced Application] For the lottery prize in problem 1 (the $10 million prize):
   a. Calculate the federal tax owed if someone collects the lump-sum payout (assume the person has no other income).
   b. Calculate the federal tax payout out per year if someone collects the monthly payouts (assume the person has no other income, and the tax table does not change over the 15 years)
   c. Which results in more total tax paid?
4 Statistical Reasoning

4.1 Sampling

Certain toothpastes may claim to get your teeth 25% whiter, or a politician may claim a certain law or program will lower unemployment. Understanding how statistical conclusions are drawn will better prepare you to examine these claims and their potential shortcomings.

The ideal way to learn characteristics of a population, or how a product or service effects a population, would be to test everyone in the population. If a teacher wants feedback on teaching style or choice of textbook for a course, it is not hard to ask each student to spend a minute or two filling out a survey.

When testing or polling an entire population is impossible or impractical, statisticians use sampling to draw conclusions. No media outlet could survey every citizen to calculate a presidential approval rating or track every single television viewer to compile ratings numbers.

How Sampling Works

Question 1: Suppose you have a large box filled with 10,000 pennies, where exactly 4,200 pennies were minted in Denver and the remaining 5,800 pennies were minted in Philadelphia. (Pennies minted in Denver have a small “D” printed under the date.) If we randomly select exactly 100 pennies from the box (after shuffling the pennies in the box thoroughly), how many pennies would we expect to have the Denver mint mark?

Answer 1: We would expect the ratio of Denver-minted pennies in the sample to match the ratio of Denver mint pennies in the entire box. Since the box contains a ratio of $\frac{4200}{10000} = 42\%$ Denver-minted pennies, we would expect the sample to contain the same ratio. With 100 pennies in the sample, we need to calculate $42\%$ of 100, or $\frac{42}{100} \times 100$, which is 42 pennies.

Would there be any surprise if only 39 of these 100 pennies were minted in Denver, or if 46 of these pennies had the Denver mint mark? Probably not. On the other hand, we might be very surprised if only ten pennies had the Denver mint mark, or if 88 pennies had the mark. While these outcomes are theoretically possible, they are so improbable that these results would lead us to believe the sample was not truly random, or that the initial assumption—that the box contained 42% Denver-minted pennies—was wrong. For example, if the Denver-minted coins were placed into the box first, and the Philadelphia mint coins were then poured on top, a handful of pennies grabbed from the box without mixing would likely yield very few Denver-minted coins.

The chance of getting no more than ten Denver-minted pennies in a random draw of 100 is about one in a trillion, and the chance of getting 88 or more Denver-minted pennies in a random draw of 100 is about one in a sextillion ($10^{21}$). The most likely outcome, drawing exactly 42 Denver-minted pennies, will happen about 8% of the time. Calculating these kinds of quantities will be left to chapter six.

The probabilities for each individual outcome follow a bell-curve, seen in figure 4.1.1. While the most likely outcome only happens 8% of the time, there is a 95% chance that the number of Denver-minted pennies will fall between 33 and 51.
These same principles apply beyond boxes of pennies. Imagine a town of 10,000 residents who have varying opinions on a new highway, and suppose that 42% of the population favors building a bypass around the town. If 100 residents are picked at random, one would expect about 42 of these random residents would favor the bypass. But the sample outcomes follow the exact same pattern as the calculations for the Denver-mint pennies. We can be 95% certain the sample will have between 33 and 51 residents who favor a government-run program.

The problem we have is that we would not know the actual percentage of residents who favor the bypass, we would only have data for the sample collected. If our sample shows that 42 of 100 randomly chosen residents favor the bypass, it may mean that 42% of the entire population holds this opinion. But if 48% of the entire population holds this opinion, there would be a 95% chance the sample would fall between 39 and 57. Since 42 falls in this interval, it is reasonable that the true percentage is 48% when the sample is 42 out of 100.

Statisticians have developed formulas to determine the most likely values for the percentage of a population holding a certain opinion or having a certain characteristic based on sample percentages. From these formulas, statisticians generate a margin of error with a confidence level. If you take a course in statistics you will learn to use these formulas.

In our example, a sample size of 42 out of 100 would fall within the 95% probability window (or a 95% confidence level) if the entire population had a percentage anywhere between 33% and 51%. This is an error of up to nine percentage points from the sample percentage of 42%, which is not considered very good. This could be written as 42% ± 9%. If a poll or study does not provide a margin of error, be suspicious of the results. The margin of error may be so large that the results are meaningless.

**Margin of Error and Confidence Level**

If we are unhappy with our nine percentage point margin of error in our poll, we can sample a larger part of the population. A larger sample size will lead to a smaller margin of error. If we imagine the extreme cases, this should be intuitive. If we sample only three people in our population of 10,000, we cannot draw any reasonable conclusion about the entire population. Similarly, if we could somehow contact 9,997 of the 10,000 population, our estimate would be incredibly accurate.

In our example, if we double our sample size to 200, then a sample percentage of \( \frac{84}{200} = 42\% \) would imply that the entire population had a percentage between 35.5% and 48.5% with 95% confidence, a margin of error of 6.5 percentage points.

| Sample Size | Confidence Level:
|-------------|----------------
| 30          | 90% 15% 18% 24% |
| 100         | 95% 8.2% 9.8% 13% |
| 200         | 95% 5.8% 6.9% 9.1% |
| 500         | 95% 3.7% 4.4% 5.8% |
| 1000        | 95% 2.6% 3.1% 4.1% |
| 5000        | 95% 1.2% 1.4% 1.8% |

Table 4.1.2

Upper Bounds on Margins of Error

Table 4.1.2 shows the largest margin of error possible, given a certain confidence level and sample size. Sample percentages far away from 50% can have smaller margins of error, while a sample percentage of 50% will have the margin of error listed in the table.

It is important to note that the margin of error is not absolute. A 95% confidence level is not 100% certainty. A 95% confidence level means that the sample could have fallen within the 5% of values away from the center of the bell curve shown in figure 4.1.1. Thus there is a chance, be it somewhat small, that the actual population percentage does not fall within the margin of error. Increasing the confidence level will increase the margin of error, but confidence can never be increased to 100% without sampling almost the entire population. Most statistical references give tables for 90%, 95%, and 99% confidence levels.

Using our example one last time, a sample size of 200 from a population of 10,000, where the sample percentage is \( \frac{84}{200} = 42\% \), gives a margin of error of 6.5 percentage points at a 95% confidence level (table
1.4.2 shows the margin of error less than 6.9%). If we lower our confidence level to 90%, the margin of error shrinks to 5.5 percentage points (less than 5.8%). If we raise our confidence level to 99%, the margin of error grows to 8.5 percentage points (less than 9.1%).

If we wish to increase confidence and decrease margin of error together, this can only be done by increasing the sample size again.

Exercises

1. A bag contains 30 red marbles and 70 blue marbles. If ten marbles are drawn randomly, what is the most likely number of red marbles drawn?

2. A small college with 1000 students has 550 women and 450 men enrolled. If 20 students are chosen at random, what is the mostly likely number of women in the group?

3. A deck of 52 cards has 13 cards that are spades. If 12 cards are drawn randomly, what is the most likely number of spades drawn?

4. A pond is filled with 600 trout where 48 of the trout have been labeled by a park ranger. If 25 fish are randomly caught, what is the most likely number of labeled trout caught?

5. A television news outlet polls 500 residents on their satisfaction with the president. At a 95% confidence level, what is the largest the margin of error could be?

6. A student newspaper polls 100 students on the quality of food at the dining hall. At a 90% confidence level, what is the largest the margin of error could be?

7. Suppose a poll with a 90% confidence level has a margin of error of two percentage points. If we raise the confidence interval to a 95% confidence level, what will this do to the margin of error?
   a. The margin of error will decrease.
   b. The margin of error will not change.
   c. The margin of error will increase.
   d. There is no way to know if the margin of error will increase or decrease.

8. A doctor’s office has randomly phoned 30 customers to estimate how many patients are satisfied with their care. The statistician has determined the margin of error is too high for the results of the phone survey to be meaningful. What should be done to fix this problem?
   a. Increase the confidence level.
   b. Increase the sample size.
   c. Retry the exact same procedure.
   d. Hire a sloppier statistician.

Is It Reasonable?

9. Nathan polls 500 residents of his home state to find out that 51% of residents support a certain policy, while 49% oppose the policy. Is it reasonable for Nathan to be sure the policy will pass at the polls the next day?

10. Olga polls 500 residents of her home state to find that 32% of residents support a certain policy, while 68% oppose the policy. Is it reasonable for Olga to be sure the policy will not pass at the polls the next day?
Advanced Applications

With a typical bell curve distribution, the margin of error decreases proportionally with the increase of the square root of the sample size. For example, increasing the sample size by a factor of four will decrease the margin of error by a factor of two, two being the square root of four. Assume the margin of error in polling a sample of 100 people is ten percentage points.

11. What sample size must be used to get an error of only two percentage points?

12. What sample size must be used to get an error of only three percentage points?
4.2 Bias

Bias occurs when the execution of a study or poll is flawed, causing the sample population to show a different characteristic than the regular population, possibly well beyond the margin of error.

In some cases, bias may be purposefully introduced into a study or poll to get a desired result. For this reason, published statistical studies are generally not conducted by the entity that has the most to gain from a certain result. A drug company will not publish studies done by itself as proof that a certain medication is effective, as the public will be skeptical that the study was not done without bias. Instead, the company will fund a study by another group that has no stake in the outcome.

In other cases, bias may be accidentally introduced into a study or poll, resulting in a false result that those performing the study may truly believe. One famous example is President Harry S. Truman holding a copy of the Chicago Tribune proclaiming his defeat to Thomas Dewey in his bid for re-election. Poll data predicted an overwhelming lead for Dewey leading into the election, but the poll data proved to be inaccurate due to a selection bias. A portion of the poll was done by telephone, which in 1948 was an item more likely to be owned by the wealthier population, who preferred the Republican candidate Thomas Dewey. [4.1] Since the Truman followers were underrepresented in the poll, the poll data was inaccurate and the newspaper relying on the poll printed an incorrect headline in hopes of being one of the first to print the story.

Sampling Bias

When sampling is not done properly, as was the case in the famous headline of 1948, selection bias becomes a concern. Have you ever heard the common lament: “How did that person get elected? Everyone I know voted for the other person.” In this case, the sample population of “everyone I know” is hardly representative of the population. Many people choose friends with similar ideological beliefs, so one’s set of friends will have a tremendous selection bias. When sampling is done based on who is easiest to ask, this is called convenience sampling.

Some of the most commonly used convenience sampling devices are website polls, public place polls, and mail-in surveys. Website polls are influenced by the characteristics of the population that visits the site. For example, a political poll on a sports website will probably have men overrepresented. Public place polls would be influenced by the audience attracted. For example, a pollster standing in a shopping mall will probably be skewed to a younger and wealthier population.

Sources of bias from convenience sampling may or may not have an impact on what is being measured. If 500 surveys are mailed out to random members of a population, asking for information on their favorite brand of peanut butter, receiving only 200 surveys in return is not likely to bias the study. The peanut butter preferences of those who return the survey are not likely to be very different from those who do not return the survey. However, if 500 surveys are mailed out to random members of a population, asking for information on stress levels, receiving only 200 surveys in return would be likely to bias the study, since those members of the population suffering more stress may be too busy to return the survey. The 200 returned surveys would be biased in favor of those in the population suffering from less stress.

Proper sampling requires a little preparation on the part of the pollster, and falls into three categories:

- **Simple random sampling** occurs when every member of the population is equally likely to be selected, and each possible subgroup of the sample size is equally likely to be selected. From a list of 1000 students, a computer program could generate 50 different random numbers from one to 1000. Those names corresponding to the numbers would be selected.
• **Systematic sampling** is a bit simpler. From the same list of 1000 students, the pollster could select every 20th name, thus the first, 21st, 41st, and 61st names get selected. The pattern continues until the entire list of 1000 is passed.

• **Stratified sampling** is a more complex form of simple random or systematic sampling, where the demographics of the sample population are manipulated to match the overall population. If a population of 1000 students is 56% men and 44% women, then a sample of size 50 would be built by randomly or systematically selecting 28 men and 22 women, which guarantees that the sample will also be 56% men and 44% women. Stratified sampling can be done with several variables, including gender, family income, race, political affiliation, region, and any other variable that might bias a study or poll if not controlled.

**Response Bias**

Even with proper sampling, the questions or survey techniques can be intentionally or unintentionally formed to lead to a certain response, creating a bias. “Wouldn’t you agree that teachers are underpaid?” or “You believe the rich should pay higher taxes, don’t you?” lead responders to agree more often than if questions are phrased more plainly, such as “Are teachers salaries too low, too high, or about right?”

Even the placement of choices within a question can affect the response. A poll conducted in a German shopping mall asked some “Would you say that industry contributes more or less to air pollution than traffic?” 45% selected traffic as the larger cause and only 32% selected industry. Others were asked “Would you say that traffic contributes more or less to air pollution than industry?” To this question, only 24% selected traffic where 57% selected industry. [4.2] Avoiding this type of bias is impossible if you are not aware of it. Organizations can avoid this problem by shuffling the order of responses in each survey.

The surroundings where a study or poll is conducted may also lead to bias. A woman conducting a poll on opinions relating to laws concerning sexual assault may get different responses than a man. Students may fill out a written survey in a classroom differently than they might fill out the same survey on an anonymous webpage from a dorm room.

**Control Groups, Blinding, and Double Blinding**

Anytime a study wants to measure the effect of something on a population, it is important to use a control group, a group of people who are not subjected to the event. For example, if a researcher wanted to see if adding extra spinach to a child’s diet makes the child stronger, simply measuring the strength of children eating extra spinach over a one-year period would not be conclusive, as almost all children, spinach-eaters and non-spinach-eaters alike, will grow stronger over a one-year period. The only way for a study to show that spinach produces stronger children is to have one group eat spinach, while studying a second group—a control group—of non-spinach-eaters. These results can then be compared, and if the group of spinach-eating children is not significantly stronger than the control group, no claim about the strength-building virtues of spinach can be made from this study. In the next section, we will discuss what we mean by “significantly stronger.”

One challenge in employing control groups is that participation in a study itself can have an effect on participants. If participants believe they are receiving medicine to heal an affliction, they may feel an improvement even if the medicine is nothing more than a sugar pill, called a placebo. The resulting positive change from believing that a fake medicine will help is called the placebo effect, which can be offset by treating a control group with a technique called blinding, where the control group is put through a similar procedure as the group being studied, except without using the ingredient or therapy being tested. Members in the control group would receive a placebo, or undergo a fake treatment regimen. The participants cannot know if they are receiving actual medication or a placebo, or the placebo would serve no purpose.
The anti-smoking medication CHANTIX reports that 44% of smoking patients studied successfully quit smoking within 9 to 12 weeks, but without knowledge of how well a control group performed (smokers who perform the same tasks but instead consume a placebo), we cannot necessarily attribute this success to the drug. For this reason, the report includes the results of the control group, where only 18% of smoking patients studied successfully quit in the same time period. [4,3] This implies that the medication was responsible for the 26 percentage point increase in success. Without the control group and proper blinding techniques, any claim would be misleading.

In some cases, even if the participants are unaware whether they are receiving a placebo or actual medicine, the doctors or researchers who carry out the study may influence the participants if they are aware which participants are in the control group. For this reason, studies like the one described above should be done with double blinding. Medications and placebos alike are coded, perhaps by a number, so that those who administer the medicine and placebos are also unaware of who is receiving what. Only those compiling the data, after the participants have completed their part of the study, know which participants were part of the control group and which were given the actual medication.

Exercises

Identify the type of sampling described in exercises 1 through 8.

1. A student stands in front of a dining hall polling students on their attitudes toward campus dining.
2. A teaching assistant selects every eighth student from a course roster of 200 students for feedback on a previous assignment.
3. A school administrator randomly selects a proportional number of students from each major to match the entire student body for a computer usage survey.
4. A recruiter places a survey on an admissions website for feedback on the attractiveness of the site.
5. The admissions office randomly selects student ID numbers from all enrolled students to select students to fill out an enrollment survey.
6. A resident assistant deals a deck of cards to the 52 new residents of the dormitory. Those that drew spade cards are selected for an orientation interview.
7. A student calls every 25th student from the campus phone directory to answer questions about religious beliefs for a statistics project.
8. An accountant selects a random sample to match proportions of students by gender and year for a survey on library usage.

In exercises 9 through 12, a control group is needed in the study. Determine if blinding, double-blinding, or no blinding is needed to get unbiased results. Explain your response in a sentence or two.

9. A professor teaching two large sections of calculus wants to know if a new textbook is more effective in teaching calculus. He uses the new text book in one of his two sections that he is teaching.
10. A cereal company adds a new ingredient to its cereal, hoping that those who eat it will feel more energetic. Those selected for the study are interviewed a few hours after eating the cereal to determine feelings of energy.
11. A researcher wishes to know if a new drug can control cravings for caffeine. Half of the participants are given the drug and the other half are given placebos. The participants are left in a residence with coffee and cola while they are observed through video to monitor caffeine consumption.

12. A counselor wants to test the effectiveness of a new technique for marriage counseling. Half of the couples coming to the counselor are helped with the new technique, while the other half are helped with a more traditional technique. One year after completing the program, the couples fill out a questionnaire rating their satisfaction with their marriage.

Is It Reasonable?

13. Reword the following question in a way that reduces response bias:
   Does the lack of variety in campus dining bother you?

14. Reword the following question in a way that reduces response bias:
   Do you resent being forced to attend Dr. Worthington’s classes to pass her course?

Advanced Applications

15. A teacher wanting to sample half of his 20 student class rolls ten 20-sided dice to randomly generate a list of ten students. Why might this technique not work?

16. Suppose a college has an enrollment that is made up of 60% women and 40% men, and the college also has 10% graduate students and 90% undergraduate students. Assume the proportion of men and women do not change between the graduate and undergraduate schools. To form a stratified sample of 50 students, how many of each group—female undergraduate students, male undergraduate students, female graduate students, and male graduate students—should be chosen?
4.3 Drawing Conclusions

Imagine you are playing a card game where you are dealt 13 cards. If all 13 cards were of the same suit, would you believe that this was a rare occurrence from a shuffled deck, or would you believe the deck was intentionally sorted? The chance that 13 cards randomly chosen from a deck of 52 cards would all be the same suit is about one in 160 billion. (In chapter six, we will examine how to calculate these probabilities.) While it is possible, the probability is so small that we are drawn to the conclusion that the deck was intentionally sorted.

Drawing conclusions in any statistical setting is very similar. If the probability of a random occurrence matching or exceeding an observed result is small enough, we conclude that the occurrence was not random. The probability is referred to as the *P-value*.

What is Small Enough?

But how small is “small enough?” In sociological and psychological settings, a *P*-value of 0.05 (or 5%, which is one chance in 20) is considered small enough. In scientific and mathematical settings, a *P*-value of 0.01 (or 1%, one chance in 100) is considered small enough. Ideally, we would love to have *P*-values like the one listed above, where one in 160 billion is about 0.0000000006%. This cannot always be done.

**Question 1:** When Charles flips a coin 1000 times, he records that heads came up 477 times and tails came up 523 times. Should Charles suspect the coin is not fair, favoring tails?

In scientific and mathematical settings, a *P*-value of 0.01 or 1% is considered rare enough to draw a conclusion that randomness alone would not account for the result. The probability of flipping exactly 477 heads and 523 tails is only 0.88%. But the standard is not measured by the probability of landing exactly on a certain number (even the probability of getting exactly 500 each of heads and tails is only a remote 2.5%). The standard is measured by the probability of getting a matching or more extreme result.

**Answer 1:** Since 477 heads is less than the 500 Charles would expect from a perfectly fair coin, a “more extreme” result would be getting even less than 477 heads. The probability of getting 477 or fewer heads in 1000 flips of a fair coin is about 7.7%. This is somewhat small, but not small enough to conclude that the coin is not fair.

**Question 2:** Dorothy flips a different coin 1000 times, and she records that heads came up 550 times and tails came up 450 times. Should Dorothy suspect the coin is not fair, favoring heads?

**Answer 2:** 550 heads is more than the 500 Dorothy would expect from a perfectly fair coin, so a “more extreme” result would be getting even more than 550 heads. The probability of getting 550 or more heads in 1000 flips of a fair coin is only about 0.09%, well below the required *P*-value of 1% for mathematical applications. Therefore, Dorothy can conclude this coin is not a fair coin.

The calculations of the probabilities in these two questions were left to a software package. Even with the knowledge from chapter six, these would be long, tedious calculations. Statistics software packages like the ones you might use if you take a course in statistics, sociology, or psychology, calculate a *P*-value from your raw data.
The Texas Sharpshooter Fallacy

Imagine firing a handful of gunshots at the side of a barn, and only after firing the gun, painting targets around each bullet hole. Someone walking by the barn might believe the shooter was an expert marksman, while the truth is that the shooter simply shot a random spray of bullets. This is an alluring trap that catches many amateur statisticians, testing for a rare occurrence only after data is gathered. One can easily examine a random sequence of events, scan for patterns, and then calculate that the probability of such a pattern occurring is small. This could lead to a false conclusion that the pattern was intentionally placed.

**Question 3:** Edgar is playing poker and is dealt five cards from a deck of 52. His cards are Q♠, 6♣, 7♥, 6♦, 6♥. Edgar figures that having three of the four sixes among his five cards is a very rare event. Later, he calculates the probability of getting three or more sixes in his hand of five cards at only 0.18%. Can Edgar assume the deck was not properly shuffled?

**Answer 3:** While 0.18% is well below the mathematical threshold of 1%, choosing the test after the hand was dealt is an unfair approach. Similar claims of unusual hands could be made with a hand containing three jacks, or four hearts, or a straight. Since Edgar is playing poker, a fair test might be to test for hands of three-of-a-kind or better, which happens about 2.9% of the time. This is unusual, but not so rare as to suspect an improper shuffle.

**Question 4:** Playing a weekend Scrabble® tournament, Felicia draws a rack of seven vowels to start her eleventh game. Should she be convinced that fate has her predestined to lose this game because the probability of drawing a rack of all vowels is only 0.17%?

**Answer 4:** Not only has Felicia’s test for the randomness of the rack been chosen after the draw, she has already drawn several racks of tiles in her previous ten games. With all those opportunities to draw an unusual rack of letters, she should not be surprised that a rack such as this would be encountered during one of her games.

Keep these principles in mind when someone tries to persuade you that some unusual but likely random event is supernatural evidence of intervention. The predictions of Nostradamus or a Bible Code depend on these statistical errors.

**Statistical Proof of God?**

Some people believe that the following argument, often labeled Intelligent Design, is scientific proof of the existence of God. The premise is that the probability that evolution through natural selection would produce such a highly complex organism, such as a human being, is so small that one must reasonably assume that a divine creator of some kind must be responsible for the creation of humankind.

This argument has the same flaw illustrated in questions three and four above, the Texas Sharpshooter Fallacy. Looking at the result first, and then defining a test for a 1% probability is not sound. Unfortunately, human beings were not around before their creation, therefore no person could set a criteria for creation before creation happened. To bolster the argument of Intelligent Design, someone would need to create a few hundred brand new earth-like planets, and monitor their evolutionary progress over the next 4.5 billion years. If none of these planets develop any complex life forms of any kind, we may have some statistical evidence of a creator. Obviously, this is not practical.

In his theological works, Emanuel Swedenborg states that without the freedom to deny a divine being, a person cannot freely choose to follow God, which is a crucial ingredient to salvation. If statistical proof of a divine creator were present, no rational person would have the freedom to deny this divine power, thus impeding any attempt at salvation. For this reason, God protects human freedom in the same way a person would protect the pupil of his or her eye. [4.4]
When is Random Too Random?

When people are asked to generate random data, the data will often have characteristics that “look” more random than typical random results. Ask someone (not in this class) to generate a list of ten random numbers from one to ten. You will likely be given a shuffled sequence of the numbers from one to ten where every number appears exactly once, such as 3, 7, 4, 2, 1, 8, 10, 6, 5, 9. In reality, the probability of ten random draws from one to ten including every number exactly once is less than 0.04%. On average, each number would appear about once per sequence over several such sequences, but the probability these averages would be realized in a single sequence is very small.

Similarly, if you ask someone to fill a box with ten randomly placed points, you will probably get ten points somewhat equally spaced in the box, where a truly random placement would have some areas more congested than others, as shown in figure 4.3.2.

In an effort to make something look random, people have a tendency to distribute events in a nearly perfect way, instead of having some clustered areas and some sparse areas. Principles as these can help determine when voting results or tax returns may be fraudulent.

Exercises

Is It Reasonable?

Identify whether the statistical conclusion is valid or invalid. Explain your answer in a sentence or two.

1. The three-digit daily lottery number matches Alex’s office number. Since the probability of this happening randomly is 0.1%, Alex concludes the lottery was fixed.

2. Clifford witnesses the first ten flips of single coin coming up heads every time. Since the probability of a fair coin coming up heads on the first ten flips is less than 0.1%, Clifford concludes the coin is double-headed or weighted to favor heads.

3. Diana deals five cards from the top of a new deck of cards, and every card is a spade. Since the probability of the first five cards from a shuffled deck of cards being the same suit is about 0.2%, Diana concludes the cards were never shuffled properly.

4. One hour into a rousing game of monopoly, Brenda rolls double sixes twice in a row. Since the probability of rolling back to back double sixes on fair dice is less than 0.08%, Brenda concludes that one of the players has switched in loaded dice.

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5. Eric and Floyd have the same golf handicap. The last three times they played a round of golf together, Floyd always managed to win, so Floyd concludes he is a better golf player than Eric.

6. On a high shelf, Gina has two identical-looking boxes. One box has red poker chips while the other has equal amounts of red and blue poker chips, thoroughly mixed. Gina reaches into one of the boxes without looking, and pulls out a handful of twenty poker chips which are all red, so Gina concludes this box only contains red poker chips. (Exact calculations for these kinds of probabilities are covered in chapter 6)
4.4 Charts and Graphs

Statistics can be displayed in a variety of ways. Frequency tables, pie charts, bar graphs and line charts are the most popular ways of displaying simple numeric data. Bar graphs and line charts can display multiple series of data. Scatter plots can display ordered pairs of numeric data.

Frequency Tables

A frequency table is the simplest of the numeric charts, listing every response and the frequency of each response. With small quantities of data, as in table 4.4.1, this choice is fine, but with larger sets of data, bar graphs and pie charts make digesting information easier.

Pie Charts

A pie chart displays numeric data in way that makes proportions easy to identify. A quick look at table 4.4.2 immediately reveals that the United States has won about half of all the Little League World Series championships, while Taiwan has won a little more than a quarter of all the championships. This information would not be immediately seen from a frequency table.

Pie charts can look cluttered if too many wedges are present, or if there are several tiny wedges. If you have more than seven or eight total categories, or more than one very narrow wedge, consider merging the smallest wedges into a single category labeled “other.” In table 4.4.2, the wedges for the countries of Venezuela (2), and Curacao (1) would have been very narrow, so these were merged into a single wedge.

Pie charts can be misleading when complete information is not available. Making the pie chart in table 4.4.3 from the data in table 4.4.1 gives the appearance that about one quarter of all presidents were born in each of Virginia and Ohio. The problem is that table 4.4.3 does not include the 14 states that have produced one president each, so only 29 of the 43 different presidents are represented in the chart. This could be fixed by including a wedge of the pie for each of the remaining 14 states, but this would create 21 total wedges, 14 of which would be very narrow. Alternatively, this could be fixed by including a large “other” wedge to represent these 14 missing states, but unfortunately this wedge would be the largest wedge on the chart, almost twice the size of Virginia’s wedge, and would dominate the picture. This would be a case where choosing a pie chart might not be the best choice; a frequency table or bar graph would be a better choice.
Bar Graphs

Bar graphs do not display proportions as well as pie charts, but they serve well in comparing sizes of multiple quantities. Furthermore, bar graphs can be enhanced in ways to display and compare multiple related quantities, a challenge the pie chart would find difficult.

The simplest bar graph only displays a single quantity attached to each label. Table 4.4.4 shows the number of named storms in the north Atlantic during the first decade of the new millennium. While we do not get the sense of proportion contained in the pie chart, it is easy to see that 2005 was by far the most active year, approximately doubling the number of storms in a typical year, while 2006 and 2009 were the least active for named storms in the decade.

```
<table>
<thead>
<tr>
<th>Year</th>
<th>Number of Storms</th>
</tr>
</thead>
<tbody>
<tr>
<td>2000</td>
<td>10</td>
</tr>
<tr>
<td>2001</td>
<td>15</td>
</tr>
<tr>
<td>2002</td>
<td>20</td>
</tr>
<tr>
<td>2003</td>
<td>25</td>
</tr>
<tr>
<td>2004</td>
<td>30</td>
</tr>
<tr>
<td>2005</td>
<td>60</td>
</tr>
<tr>
<td>2006</td>
<td>50</td>
</tr>
<tr>
<td>2007</td>
<td>40</td>
</tr>
<tr>
<td>2008</td>
<td>35</td>
</tr>
<tr>
<td>2009</td>
<td>30</td>
</tr>
</tbody>
</table>
```

Bar graphs can display multiple quantities either using a side-by-side approach or a stacking approach. The side-by-side bar graph is best for allowing comparisons of values within a category. Table 4.4.6 shows that the National Basketball Association (NBA) is the league where the proportion of Eastern time zone teams is closest to that of the Central time zone.

```
<table>
<thead>
<tr>
<th>League</th>
<th>Eastern</th>
<th>Central</th>
<th>Mountain</th>
<th>Pacific</th>
</tr>
</thead>
<tbody>
<tr>
<td>NFL</td>
<td>15</td>
<td>20</td>
<td>10</td>
<td>5</td>
</tr>
<tr>
<td>MLB</td>
<td>10</td>
<td>25</td>
<td>15</td>
<td>5</td>
</tr>
<tr>
<td>NBA</td>
<td>20</td>
<td>30</td>
<td>20</td>
<td>10</td>
</tr>
<tr>
<td>NHL</td>
<td>5</td>
<td>5</td>
<td>20</td>
<td>30</td>
</tr>
</tbody>
</table>
```

The stacked bar graph is best for allowing comparisons of totals and, in some cases, comparing the same group across categories. Table 4.4.6 shows that all the sports leagues have about the same number of teams, with the National Football League (NFL) slightly above the others. By comparing the thickness of the bands, the table shows that the National Hockey League (NHL) has the smallest number of Central time zone teams and the largest number of Mountain time zone teams. It is a little trickier to notice that the bands of Eastern time zone teams are identical for the NFL and NHL since the bands occur at differing heights. A little subtraction (32 – 15 = 17 for the NFL, and 30 – 13 = 17 for the NHL) shows these bands are identical.
A bar graph can be misleading if the scale on the vertical axis is not appropriate for the application. Table 4.4.7 exaggerates the differences between male and female SAT math scores by making one bar almost ten times larger than the other, while table 4.4.8 disguises the temperature differences between cities by allowing the temperature scale to include temperatures never seen on the planet.

A bar graph can also be misleading if the bar is made to appear disproportionately large. In table 4.4.9, we see that Californians pay about twice the gasoline tax paid by those in Wyoming, but since the icon is both twice as tall and twice as wide, the visual impact is four times larger. Since the image used is a picture of a three-dimensional object, the bag appears to hold eight times the quantity.

Line Charts

Line charts can be used in the same contexts as bar graphs. The points indicate the quantities instead of the bars, and the points are then connected by lines. Since the points are connected, line charts are best used where values in between categories make sense. Data categories by state, for example, would best be done by a bar graph. Data categories by time, on the other hand, are more appropriate for line graphs.

But not all time scales are best served by a line graph. In table 4.4.10, the scale is by year, but there is no hurricane season between 2004 and 2005 where the number of named storms is between 15 and 27. Table 4.4.11 shows a good use of a line chart, as we could reasonably approximate the population of Philadelphia using the line between two data points. It is reasonable to believe that the metropolitan area for Philadelphia would have a population of about 6 million in 1995.

Exercises

1. A pizza delivery driver recorded the different types of single-topping pizzas delivered in one evening. Construct a frequency table, a pie chart, and a bar graph for the data collected:
   Pepperoni, Sausage, Pepperoni, Pepperoni, Onions, Mushrooms, Mushrooms, Pepperoni, Sausage

2. A random group of students were asked in what field they were majoring. Construct a frequency table, a pie chart, and a bar graph for the data collected:
3. According to Table 4.4.11, around what year did the population of the Philadelphia metropolitan area reach five million?

4. According to Table 4.4.11, around what year was the population of the Philadelphia metropolitan area twice as big as the population of the city proper?

5. Using table 4.4.12, about what fraction of the area of world’s oceans is covered by the Pacific Ocean?

6. Using table 4.4.12, about what fraction of the area of world’s oceans is covered by the Atlantic Ocean?

Is It Reasonable?

For exercises 7 through 12, determine which type of graphic would be best to display the data (frequency table, pie chart, bar graph, or line chart), and display the data with this graphic.

7. Revenue sources for a college: Tuition and Fees: $36 million; Endowment Payout: $15 million; Contributions and Gifts: $6 million; Federal Aid: $12 million; Other Sources: $3 million.

8. Total occupied condominium units by year: 2011: 26 units; 2012: 45 units; 2013: 82 units; 2014: 103 units.

9. Favorite fast foods for a group of eight students: pizza, burgers, burgers, tacos, pizza, burgers, chicken, pizza.

10. Population of Canada’s largest metropolitan Areas: Toronto: 5.5 million; Montréal: 3.8 million; Vancouver: 2.3 million; Ottawa: 1.2 million; Calgary: 1.2 million; Edmonton: 1.1 million.


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Iggy’s Ice Cream Shoppe was tracking sales of their three new flavors of ice cream over a holiday weekend. On Friday, Iggy sold 15 scoops of Merry Mango, 12 scoops of Luscious Lime, and 16 scoops of Keen Kiwi. On Saturday, Iggy sold 17 scoops of Merry Mango, 15 scoops of Luscious Lime, and 14 scoops of Keen Kiwi. On Sunday, Iggy sold 12 scoops of Merry Mango, 14 scoops of Luscious Lime, and 9 scoops of Keen Kiwi.

13. Construct a stacked bar graph for the data above.

14. Construct a side-by-side bar graph for the data above.
4.5 Scatter Plots and Correlation

A scatter plot is a graph used to display pairs of data. In table 3.5.1 we have two pieces of data for each team: the number of field goals allowed and the number of wins. The values can be plotted on a grid with field goals along the x-axis and wins on the y-axis. This relationship is not required to be a function. In this example, both the Houston Texans and Detroit Lions allowed 25 field goals during the season, but the Lions won no games while the Texans won eight games.

Correlation

The arrangement of points in table 4.5.1 does not suggest a relationship between the two variables, field goals allowed and wins. There are teams that allowed many field goals but did not win a lot (Seattle) and teams that did win a lot (Indianapolis), and there are teams that allowed few field goals and did not win a lot (Cleveland) and teams that did win a lot (Tennessee). This suggests that we cannot predict a team’s success based on the number of field goals the team allowed. A pattern where the increase of one variable is not connected with a change of another variable shows no correlation.

The arrangement of points in table 4.5.2 tells a different story. There appears to be a connection between how many points a team scores and how many wins a team gets. This is not a strict rule, as we see that Detroit scored more points than Cleveland, yet Cleveland had more wins. However, there is definitely a pattern where teams with more points tend to be the teams with more wins. A pattern where the increase of one variable coincides with the increase of another is called a positive correlation. When the pattern is followed closely, as the example in table 4.5.2, this is considered a strong correlation.

The arrangement of points in table 4.5.3 suggests a pattern, but not as strongly as table 4.5.2. There is a trend visible, where throwing more interceptions would predict fewer wins. A pattern where the increase of one variable coincides with the decrease of another is called a negative correlation. When the pattern is visible but not followed closely, as the example in table 4.5.3, this is considered a weak correlation. One noticeable exception to the trend is the Dallas Cowboys, who threw 33 interceptions, yet managed to win nine games. A data point that breaks the pattern followed by most of the other points is called an outlier.

Confounding Variables

When drawing conclusions about correlations, watch out for other variables that might better explain the correlation. One can easily be fooled into believing an apparent correlation exists between two quantities when in actuality a third hidden variable explains the correlation instead. Even worse, a statistician with an
agenda could make two unrelated quantities appear to have a correlation when no true correlation exists when accounting for a third hidden variable.

Consider table 4.5.4, showing the heights and average weekly hours of video game playing for a set of college students. There appears to be a weak correlation that is still statistically significant, meaning that the probability of getting random points to match a trend of this strength is still less than 5%. But does it make sense that taller students are likely to play more video games?

Separating men from women tells a very different story. In table 4.5.5 there is no trend indicating that taller women play more video games than shorter women, and in table 4.5.6 there is no trend indicating that taller men play more video games than shorter men.

What is clear is that men tend to play more video games than women and men also tend to be taller than women. The weak positive correlation seen in table 4.5.4 between height and hours of game playing can be fully explained by breaking down the results by a third variable, namely gender.

### Correlation and Cause

Another pitfall for statisticians studying correlation is cause. The fact that two quantities may be correlated does not imply that one must cause the other.

Imagine a street-side ice cream vendor tracking weekday sales for three weeks in July. If that vendor were to check in with the nearby emergency room, he could gather data on cases of heat stroke. The scatter plot of this data might look like table 4.5.7, which shows a pretty strong correlation between the two variables.

In this example, we hope the vendor would not believe that his selling more ice cream cones actually causes there to be more cases of heat stroke. (Quick! Shut down all the ice cream vendors for public safety!) Instead, we suspect there is a third variable, average daily temperature, which would be blamed for the higher number of both heat stroke cases and ice cream sales.

Other examples may be less clear. If a study shows a negative correlation between eating bananas and pancreatic cancer rates (no such study exists to my knowledge; this is just an example), it means that people who eat more bananas are less likely to get pancreatic cancer. This does not necessarily mean that increasing banana consumption will reduce your likelihood of developing pancreatic cancer. It could be that some hidden variable, like the daily temperature in the example above, is the underlying cause behind making people crave bananas and lowering pancreatic cancer rates. Unfortunately, this distinction is often lost, as those people who would profit from banana sales would be quick to present these results, hoping
that the public will falsely believe that it is banana consumption that will cause the reduction in this cancer
rate. While there may be a cause and effect relationship between correlated variables, the fact the variables
are correlated is not enough to prove a cause-and-effect relationship.

Exercises

For each of the graphs in exercises 1 through 6, identify whether there is a positive correlation, negative
correlation or no correlation. For those with a correlation, specify whether the correlation is strong or
weak.

1.  
2.  
3.  
4.  
5.  
6.  

Is It Reasonable?

In exercises 7 through 12, state whether you believe the variables would be positively correlated, negatively
correlated or not correlated.

7. Average attendance for a major league baseball team over one season and total wins for a major league
   baseball team over the same season.

8. Age of an adult cell phone user and number of text messages sent by an adult cell phone user.

9. Weight of a motor vehicle and fuel efficiency of a motor vehicle.

10. Grade point average for a college student and shoe size for a college student.

11. Route number for a highway and traffic volume for a highway.

12. Selling price for a property and lot size for a property.

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13. A researcher discovers a negative correlation between number of text messages sent by adults and
    frequency of heart attacks in adults. Certainly sending more text messages is not likely to have any
    effect on one’s likelihood of suffering a heart attack. What confounding variable would likely explain
    this correlation?

14. A researcher wants to prove that listening to classical music before a mathematics test improves one’s
    performance on the test. After recording grades on a standardized test, the researcher interviews the
    students on their music listening habits and finds a positive correlation between the variables. How
    would this study need to be changed to prove that listening to classical music will actually cause an
    improvement in test performance?
4.6 Averages

The object of many studies is to capture what a “typical” case might look like. How cold is a typical February in Atlanta? How much time is typically needed before headache pain will subside after taking a pain reliever? How many minutes does a typical student talk on a cell phone each month? There are three common calculations that are used to measure “typical” behavior given a set of data: mean, median, and mode.

Definitions

Table 4.6.1 shows the data from table 4.5.4 organized by response (with information on heights removed). If we want to know how many hours a “typical” college student spends playing computer games, we can look at the three quantities.

**Question 1:** What is the mode of the data in table 4.6.1?

The *mode* of a data set is the result that is most common.

**Answer 1:** According to table 4.6.1, 11 of the 30 students claimed that they do not play computer games, far more than any other single response. In this sense, zero is considered the most “typical” answer.

**Question 2:** What is the median of the data in table 4.6.1?

The *median* of a data set is the result where exactly half of the responses are above the result and half of the responses fall below the result. When an odd number of data points are gathered, there will always be a middle number when the data is sorted by response value. For example, with data, 2, 5, 6, 9, 17, the middle number is six, so this is the median. When an even number of data points are gathered, there will be two middle numbers, and the median is half-way between these. For example, with data, 2, 5, 6, 9, 17, 22, the two middle numbers are six and nine, so the median is the number is half-way between, \( \frac{6+9}{2} = 7.5 \).

**Answer 2:** Sorting the values in table 4.6.1 gives the list:

0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 2, 2, 2, 3, 3, 3, 4, 4, 4, 5, 5, 6, 6, 8, 15.

There are 30 responses, an even number, and so we have two middle numbers. In this case, both middle numbers are twos (underlined) so the median value is two. In this sense, two is considered the most “typical” answer.

**Question 3:** What is the mean of the data in table 4.6.1?

The *mean* of a data set is an average where every data value is given equal weight. One could imagine placing pennies on a ruler at various marks representing responses, one penny for each response. The mean would be the value where the ruler would balance. It is calculated by adding all the responses and dividing by the number of responses. With data, 2, 5, 6, 9, 17, the mean would be \( \frac{2+5+6+9+17}{5} = 7.8 \). With data, 2, 5, 6, 9, 17, 22, the mean would be \( \frac{2+5+6+9+17+22}{6} = 10.1666… \) is customary to round means to one decimal place beyond the precision of the data. Thus with integer data, the mean would be rounded to the nearest tenth, giving 10.2 for the mean. In statistical studies, the Greek letter \( \mu \) (pronounced “myoo”) represents the mean value.

A mean should be rounded to one decimal place beyond the precision of the data.
Answer 3: The sum of the 30 responses in table 4.6.1 is 78. Dividing by 30 gives a mean of \( \mu = 2.6 \) (which happens to already show accuracy to one decimal place). In this sense, 2.6 is considered the most “typical” answer.

With a normal distribution, such as the distribution for the probabilities of flipping a number of heads on 1000 coin tosses, the three calculations, mode, median, and mean, have the same value. In figure 4.6.3, we see these measurements are each 500. This creates no ambiguity when we say that the “average” is 500, since no matter which calculation is used, the answer is the same.

Many statistical calculations with a large sample size fall into a normal distribution, such as weights of newborn babies and SAT scores of college-bound students. In these cases, the choice of whether to use a mode, median, or mean is irrelevant, since they are the same. But when data does not have a normal distribution, which calculation is the most descriptive?

Using the Mean

Imagine a satisfaction survey where customers rate service on a scale from 1 (poor) to 5 (excellent). Figure 4.6.4 shows hypothetical results from two surveys. In both surveys, four is the most common answer, the mode. Four is also the median, being the middle value of both surveys when sorted by response value:

- Left: 1, 1, 2, 2, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3, 3, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 5, 5, 5, 5, 5
- Right: 2, 2, 3, 3, 3, 3, 3, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 5, 5, 5, 5, 5, 5, 5, 5, 5

The two graphs seem to tell very different stories about customer service, despite the fact that the modes and medians are identical. However, the means for these two sets of data are quite different. The sums of the scores from the left chart give a sum of 120 and thus an average of \( \frac{120}{35} = 3.4 \) (when rounded to one decimal place), while the sums of the scores from the right chart give a sum of 143 and thus an average of \( \frac{143}{35} = 4.1 \) (when rounded to one decimal place). In this case the mean tells the best story.

The mean will often be the best measure of an average case since every data point is used in the calculation. This is especially important when the variety of values is limited, such as a survey where responses are limited to a small range of values.

Using the Median

There are a few instances where the mean may not be the best measure of average. If numeric values can differ by multiple orders of magnitude, a mean can be misleading as a few very large or very small numbers can influence the mean calculation. Imagine a group living in a neighborhood where household incomes are (in thousands) $35, $40, $50, $55, $65, $70, $75, $80, $100, and $110. The mean of these incomes is $68,000, which is a reasonable measure of the average income. If LeBron James were to move into the neighborhood, with his $19 million salary and an estimated $53 million endorsement income (as of 2014) [4,5], his $72 million income would raise the average to $6,607,273, which is not representative of the neighborhood income.
The median income before James moved in would be half way between the two middle numbers ($65 and $70 thousand), giving $67,500. After James moved in, the middle number would now be $70,000, as there are now eleven income numbers, five incomes greater and five incomes less than this number. $70,000 would be a better measure of a typical income in this neighborhood.

Using the Mode

The short answer here is never. The mode gives the most popular value, but since so little information goes into the calculation of the mode, the information in the mode has very little value.

The mode is the most misleading when values take on such a wide variety that the most popular value is still a very small percent of the population. If we examine the yearly consumption of coffee for adult U.S. residents in cups, most values would be in the hundreds (between one every three days to three every day), but the number of citizens drinking exactly 254 cups in a year would be very small. The number drinking no coffee during the year (estimated by ncausa.org) is only 23%, but this is larger than any other individual value, thus zero would be the mode for this statistic.

The mode would be considered typical if most values, more than half, fall into one value. However, in this case, the median would match the mode, so the mode offers no additional information.

The mode may not be a unique value. In cases where more than one value appears the same number of times and more than any other value, each of these values is considered a mode. With a set of data such as: 1, 2, 3, 3, 4, 5, 6, 7, 8, 9, 10; both three and seven are considered modes. In cases where no value is repeated, every value is a mode, so this data cannot be summarized by a mode.

One place where the term mode creeps into the conversation of statisticians is in describing distributions of data. A normal distribution tends to have values clustered around a single value, where mean, median, and mode all meet. A bimodal distribution has two distinct clusters of values, not necessarily the same size. These distributions tend to surface when a population has two distinct types, such as weights of pennies, where pre-1983 pennies are mostly copper and post-1983 pennies are mostly zinc, which is lighter than copper.

Exercises

In exercises 1 through 4, calculate the mode, median, and mean of the data set.

1. Skill proficiency scores: 0, 2, 1, 2, 2, 3, 2, 0, 0, 1, 2, 2, 3, 2.

2. Satisfaction survey results: 4, 2, 5, 3, 4, 2, 5, 3, 4, 4, 2, 3, 3, 1, 4.

3. Areas of New England states:
   - Connecticut: 4,845
   - Maine: 30,862
   - Massachusetts: 7,840
   - New Hampshire: 8,968
   - Rhode Island: 1,045
   - Vermont: 9,250
4. Populations of National League Central markets as of 2010:
   - Chicago Cubs: 9,525,000
   - Cincinnati Reds: 2,134,000
   - Houston Astros: 5,628,000
   - Milwaukee Brewers: 1,544,000
   - Pittsburgh Pirates: 2,356,000
   - Saint Louis Cardinals: 2,804,000

Is It Reasonable?

In exercises 5 through 8, state which is more representative of a typical instance, median or mean.

5. Marathon times in a public city marathon.
6. Numbers of car accidents for individuals in the last year.
7. Numbers of courses taken over one year for individual full-time college freshmen.
8. Distances traveled by college students over the summer break.

Advanced Applications

9. Calculate the median where 35% of values are 10, 20% of values are 11, 30% of values are 12, and 15% of values are 13.

10. Calculate the mean where 35% of values are 10, 20% of values are 11, 30% of values are 12, and 15% of values are 13. These types of calculations are called weighted averages and will be studied in section 6.6.
4.7 Review Exercises

1. Suppose that 45% of all college students participate in college athletics of some kind:
   a. If you sampled 20 random college students, how many would you expect to participate in college athletics?
   b. If your sample of 20 students had eight athletes, would you be surprised? If yes, what conclusion would you draw?
   c. If your sample of 20 students had 17 athletes, would you be surprised? If yes, what conclusion would you draw?

2. Identify problems with the design of the following study:
   John Q. Student suspects a link between video game playing and mathematical ability.
   - He selects six students from his floor in his dormitory known for playing lots of video games.
   - He asks each student to answer a few math problems.
   - He then asks “A lot of people believe playing video games is a waste of time, but would you say that playing video games has increased your spatial intelligence and thus can be helpful with your math ability?” and marks their response.

3. Describe why the following reasonings are not valid:
   a. A group of twenty seniors went to Atlantic City to play the slot machines. On her first try, one of the seniors won a jackpot whose probability was only 0.4%. The group concluded that this one particular slot machine was rigged.
   b. A statistics student at a college Scrabble® tournament discovers a positive correlation between average Scrabble® scores and college GPA. The student concludes that students who become better Scrabble® players will raise their GPA.

4. Construct a frequency table, pie chart, and bar graph for the following lists:
   a. Second graders favorite colors:
      red, green, red, blue, blue, orange, red, blue, purple, blue, orange, green
   b. World Cup champions:
      Uruguay, Italy, Italy, Uruguay, Germany, Brazil, Brazil, England, Brazil, Germany, Argentina, Italy, Argentina, Germany, Brazil, France, Brazil, Italy, Spain, Germany

5. Answer the following based on figure 4.7.1.
   a. How many students have a GPA above 3.5?
   b. How many students have a GPA between 2.0 and 3.0?
   c. How many students have a GPA below 2.5?
   d. What percentage of students have a GPA above 3.0?

6. Calculate the mode, median, and mean for the following data sets, and determine which is most representative of what is “typical.”
   a. GPA’s of students:
      4.0, 3.8, 3.7, 3.4, 3.1, 2.9, 2.8, 2.6, 2.6, 2.3, 1.8
      93, 87, 86, 70, 59
   c. Home values on Main Street (in thousands):
      150, 170, 220, 220, 230, 250, 260, 280, 300, 320, 2200, 3250

Table 4.7.1

GPA’s of students
4.8 Project: Correlation Significance

In chapter one, we did a project on *linear regression*. The regression formula will always find a best fitting line equation, whether there is a significant link between variables (strong correlation) or not (no correlation). In addition to finding this trendline equation, most of these same software packages can also calculate the *Pearson Correlation Coefficient*, which measures how significant a correlation pattern is.

A correlation coefficient of zero indicates no correlation, while a coefficient of 1 (positive) or −1 (negative) indicates a perfect coefficient. If the value of the coefficient is far enough away from zero, compared to a correlation from typical random points, the correlation is said to be significant. The more points in the data set, the less strong the correlation must be in order to be significant, because random points are less likely to line up when you have more points.

Here are the instructions from the first project again, requiring the use of Microsoft Excel®, except that we want to see the correlation coefficient instead of the trendline equation.

A. Open Microsoft Excel® (or similar program)

B. In the left-most column, enter the *x* values for your points, and in the second column from the left, enter the corresponding *y* values for your points.

   For example, if your points are (2,10), (5,8), (10,5), and (12,6), your spreadsheet should look like the first figure to the right:

C. Select the cells containing you data (drag from one corner to the other).

D. From the insert menu, insert a chart. Choose the “XY (Scatter)”. Select the figure with no lines, just points.

E. Click on any of the data points (this will select them all), and then right-click on any of the data points and choose “Add Trendline…”.

F. In the trendline options, choose “Linear” (it is probably the default choice, since it is the most common), and instead of checking the “Display Equation on chart” box, check the box next to “Display R-squared value on chart”.

The value displayed is equal to $r^2$. To find $r$, take the square root of this number. Use the positive square root if the trendline has a positive slope, and use the negative square root if the trendline has a negative slope. In this example, the trendline has a negative slope, so we have $r = -0.929$.

To determine whether this value is significant, compare the absolute value (remove the negative sign if it is negative, otherwise, just use the value) of $r$, to the value in table 4.8.1. If the value is higher than the number on the chart, the correlation is significant to the appropriate confidence level. As we learned in chapter three, we never get 100% confidence, there is always a chance, small it may be, that random points will line up.
Use the instructions from the previous page to do the following:

1. Create four random points using pairs of 20-sided die rolls (if you do not have a 20-sided die handy, use Excel’s RANDBETWEEN function to create random numbers from 1 to 20). Then graph these points, and find the value of the correlation coefficient. Finally compare this value to table 4.8.1 to check for significance.

2. Do the same process as problem 1, but use ten random points instead of four. Which value of $r$ was higher? Which value of $r$ was closer to being significant?

3. Select ten countries, states, or provinces and create ten data points by finding the population of the country, state, or province and the population of the largest city in the country, state, or province. For example, using the 2010 U.S. census for Illinois (largest city is Chicago), one point could be (12,830,632, 2,695,598). Be sure to use the city proper, not the metropolitan area, as metropolitan areas can overlap state and province boundaries (and in some cases, national boundaries). Finally, calculate the correlation coefficient and test for significance.

4. Do the same process as problem 3, but use the population of the capital city instead of the largest city. Which had the stronger correlation, largest city or capital city? Why might this be the case?

<table>
<thead>
<tr>
<th>Sample</th>
<th>90%</th>
<th>95%</th>
<th>99%</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0.900</td>
<td>0.950</td>
<td>0.990</td>
</tr>
<tr>
<td>5</td>
<td>0.805</td>
<td>0.878</td>
<td>0.959</td>
</tr>
<tr>
<td>6</td>
<td>0.729</td>
<td>0.811</td>
<td>0.917</td>
</tr>
<tr>
<td>7</td>
<td>0.669</td>
<td>0.754</td>
<td>0.875</td>
</tr>
<tr>
<td>8</td>
<td>0.621</td>
<td>0.707</td>
<td>0.834</td>
</tr>
<tr>
<td>9</td>
<td>0.582</td>
<td>0.666</td>
<td>0.798</td>
</tr>
<tr>
<td>10</td>
<td>0.549</td>
<td>0.632</td>
<td>0.765</td>
</tr>
<tr>
<td>12</td>
<td>0.497</td>
<td>0.576</td>
<td>0.708</td>
</tr>
<tr>
<td>15</td>
<td>0.441</td>
<td>0.514</td>
<td>0.641</td>
</tr>
<tr>
<td>20</td>
<td>0.378</td>
<td>0.444</td>
<td>0.561</td>
</tr>
<tr>
<td>25</td>
<td>0.337</td>
<td>0.396</td>
<td>0.505</td>
</tr>
<tr>
<td>30</td>
<td>0.306</td>
<td>0.361</td>
<td>0.463</td>
</tr>
<tr>
<td>40</td>
<td>0.264</td>
<td>0.312</td>
<td>0.403</td>
</tr>
<tr>
<td>50</td>
<td>0.235</td>
<td>0.279</td>
<td>0.361</td>
</tr>
<tr>
<td>60</td>
<td>0.214</td>
<td>0.254</td>
<td>0.330</td>
</tr>
<tr>
<td>80</td>
<td>0.185</td>
<td>0.220</td>
<td>0.286</td>
</tr>
<tr>
<td>100</td>
<td>0.165</td>
<td>0.197</td>
<td>0.256</td>
</tr>
</tbody>
</table>

Table 4.8.1
Minimum significant values for the absolute value of the Pearson correlation coefficient
Chapters 5 to 8

are not printed with this packet since they are not covered in Math101 at Bryn Athyn College. This keeps your costs down.

The content is available at ns.faculty.brynathyn.edu
## 9 Appendices

### 9.2 Answers to Odd Exercises and All Review Problems

<table>
<thead>
<tr>
<th>Section 0.1</th>
<th>1. 37</th>
<th>3. 29</th>
<th>5. $\frac{187}{12}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Section 0.2</td>
<td>3. $4b + 12$</td>
<td>5. $3ab + 7a - 6b$</td>
<td>7. $-(\frac{1}{2})h + (\frac{3}{2})k - 2$</td>
</tr>
<tr>
<td>9. $-ab + 2a - 3b + 6$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Section 1.1</td>
<td>1. $x - 15$</td>
<td>3. $(\frac{1}{2})q + 20$</td>
<td>5. $180 - 2p$</td>
</tr>
<tr>
<td>11. yes</td>
<td></td>
<td></td>
<td>7. $x^2 + x$</td>
</tr>
<tr>
<td>13. $\frac{3}{2}h + 3$</td>
<td></td>
<td></td>
<td>9. $6z - 12$</td>
</tr>
<tr>
<td>Section 1.2</td>
<td>1. $x = 2$</td>
<td>3. $x = 5$</td>
<td>5. no solution</td>
</tr>
<tr>
<td>11. $k = 6$</td>
<td></td>
<td></td>
<td>7. $p = 2q - 3$</td>
</tr>
<tr>
<td>9. no</td>
<td></td>
<td></td>
<td>9. No</td>
</tr>
<tr>
<td>Section 1.3</td>
<td>1. 4 and 10</td>
<td>3. 7 souvenirs</td>
<td>5. 4 touchdowns, 3 field goals</td>
</tr>
<tr>
<td>9. no</td>
<td></td>
<td></td>
<td>7. 6 quarters</td>
</tr>
<tr>
<td>11. yes</td>
<td></td>
<td></td>
<td>11. yes, she will need about $174.</td>
</tr>
<tr>
<td>13. $14$ $20$-bills, $12$ $1$-bills</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Section 1.4</td>
<td>3. (a) and (c) are functions</td>
<td>5. $4$, $-5$</td>
<td></td>
</tr>
<tr>
<td>9. 105,000 square miles</td>
<td></td>
<td></td>
<td>7. $-\frac{1}{5}$, $2$</td>
</tr>
<tr>
<td>15. $4h + 5$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Section 1.5</td>
<td>1. 2</td>
<td>3. $-\frac{1}{4}$</td>
<td>5. $\frac{2}{5}$</td>
</tr>
<tr>
<td>7. slope: $\frac{1}{2}$, y-intercept: $-1$</td>
<td></td>
<td></td>
<td>17. not a line</td>
</tr>
<tr>
<td>19. infinitely steep (line is vertical)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Section 1.6</td>
<td>1. $y = 2x + 5$</td>
<td>3. $y = -4x + 11$</td>
<td>5. $y = (\frac{1}{2})x + 2$</td>
</tr>
<tr>
<td>9. $y = 6x + 2$, $$56$, 18 pizzas</td>
<td></td>
<td></td>
<td>7. $y = -(\frac{1}{2})x - 1$</td>
</tr>
<tr>
<td>11. $y = 6.5x + 4$, 23.5 inches, about 7.1 hours</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>13. $W(r) = 40r - 300$, 60 grams, 7.5 centimeters</td>
<td></td>
<td></td>
<td>15. no</td>
</tr>
<tr>
<td>19. $C(h) = 3h$</td>
<td></td>
<td></td>
<td>17. Yes</td>
</tr>
<tr>
<td>Section 1.7</td>
<td>1: a) $11 - p$</td>
<td>b) $3x + 11$</td>
<td>c) $q = 4$</td>
</tr>
<tr>
<td>2: a) $r = 1$</td>
<td>b) $x = 10r$</td>
<td>c) $g = 4$</td>
<td>d) $g = \frac{3}{2} - \frac{1}{2} + 1$</td>
</tr>
<tr>
<td>3: a) 6 nickels</td>
<td>b) 5, 6, 7, and 8</td>
<td>c) not a function</td>
<td>d) function</td>
</tr>
<tr>
<td>4: a) function</td>
<td>b) not a function</td>
<td>c) not a function</td>
<td>d) function</td>
</tr>
<tr>
<td>5: a) 7</td>
<td>b) $-5$</td>
<td>c) $\frac{1}{2}$</td>
<td>d) $3q - 2$</td>
</tr>
<tr>
<td>6: a) line, 3, 7</td>
<td>b) not a line</td>
<td>c) line, $-1, 8$</td>
<td>d) line, $\frac{2}{3}$, 0</td>
</tr>
<tr>
<td>f) line, 9999, $-2000$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7: a) $y = (\frac{1}{2})x + 1$</td>
<td>b) $y = 2x - 1$</td>
<td>c) $y = -4x + 3$</td>
<td>d) $y = (\frac{1}{2})x - \frac{10}{3}$</td>
</tr>
<tr>
<td>8: a) $y = 0.08x + 140$</td>
<td>b) $$364$</td>
<td>c) 3250 copies</td>
<td></td>
</tr>
<tr>
<td>Section 2.1</td>
<td>1. 2 and 3 significant figures, 790 square meters</td>
<td>5. 44,900</td>
<td></td>
</tr>
<tr>
<td>3. tenths and hundredths, 104.8 ounces</td>
<td></td>
<td></td>
<td>7. 30,000</td>
</tr>
<tr>
<td>9. 240</td>
<td>11. 1400</td>
<td>13. no</td>
<td>15. 0.00</td>
</tr>
<tr>
<td>Section 2.2</td>
<td>1. 6620; 0.00031; 1,370,000,000,000</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

114
3. $3.5 \times 10^5, 8.7 \times 10^5, 1.3570 \times 10^2, 9.82 \times 10^{34}$
5. $8.7 \times 10^9$
7. $1.63 \times 10^4$
9. $4.56 \times 10^1$
11. $2.71 \times 10^5$
13. $2.894 \times 10^8$
15. $10^{10}$
17. $-1.0 \times 10^{-1}$

Section 2.3
1. 211 ounces
3. 25 hours
5. 1.5 inches per day
7. 664,000 square miles
9. 130 pounds per square inch
11. 80.55 pesos
13. 155 meters
15. 35 miles per gallon
17. a tourist’s t-shirt
19. a professional soccer player

Section 2.4
1. 0.008 s
3. 42.195 km
5. 55,000 cm
7. 0.11 km
9. $6 \times 10^{-3}$ μs
11. $6.0 \times 10^{-30}$
13. 41°C
15. 12°F
17. grams
19. milliseconds
21. −320°F

Section 2.5
1. 22
3. 5.4 million
5. 45%
7. 62%
9. 38% more
11. 58% less
13. 105 messages
15. $1.12$
17. 33% more
19. yes
21. no
23. $390

Section 2.6
1. New York 100, Los Angeles 154, Chicago 123, Dallas 85, Miami 177, Washington 138
3. 50.4%
5. $170,000
7. 4.0% and 2.9%, 1997 to 1998 is higher
9. the house purchased in 1991
11. 81°
13. no, it went down in 1949, 1955, and 2009, and could go down again.
15. 87.2% and 61.2%, housing prices have increased more

Section 2.7
1: a) 3
b) 340
c) 1900
d) 0.1429
e) 800
f) 10.7
g) 5.0
h) 740,000
i) 13.0
j) 1.41 kg/cm³
2: a) $3.5 \times 10^{-7}$
b) $3.46 \times 10^8$
c) $7.2 \times 10^{13}$
d) $4.4 \times 10^3$
e) $1.82 \times 10^{-3}$
f) $7 \times 10^5$
g) $2.3 \times 10^5$
h) $1.001 \times 10^{-2}$
i) $3.00 \times 10^5$
j) 84,000,000,000 or $8.4 \times 10^{10}$ mm³
3: a) 120 yd
b) 86,400 min
c) 20 cm
d) 454 g
e) 0.8505 kg
f) $32 \frac{6}{7}$
g) ¥14,700
h) 1.32 ft²
i) 84,000,000,000 or $8.4 \times 10^{10}$ mm³
j) 1.41 kg/cm³

Section 3.1
1. $208
3. $1820
5. About $216
7. About $5692
9. no for sour cream (it will spoil before all used); yes for bottled water

Section 3.2
1. $420
3. $777.14
5. $1877.61
7. 4.289%
9. no
11. 13.86 years

Section 3.3
1. $36,677.46
3. $20.14
5. $1199.10
7. $13,574.77
9. no, anyone could borrow money and profit simply by saving the money at the bank.
11. $619.11

Section 3.4
1. $1011.80 – $1010.76 = $1.04 saved on the monthly payment by buying the lower rate.
3. (1) $340.18 (2) 345.44. The payment is $5.26 lower by using the 1% interest rate.
5. no, with a higher interest rate, the principal is paid off more slowly at the start, increasing the amount of interest paid over the loan by a much larger amount.
Section 3.5

1. lump-sum  3. flat  5. $35,175.75  7. $10,662.50  9. no

Section 3.6

1: about $14,668
2: a) $5110  b) $624  c) $1560  d) $600
3: a) $118  b) $216.60  c) $310.68  d) $539.94  e) $951.33
4: a) $3734.98  b) $3889.98  c) $136.10
5: a) 9%  b) 8.3%  c) 7.25%  d) 6.18%
6: a) $297.02  b) $1342.05  c) $92.63
7: a) $19,775.75  b) $164,552.90  c) $7162.50
8: a) Interest can be earned on previous interest payments with compound but not simple interest.
    b) With a progressive tax, the rich pay a higher rate than the poor.
    With a regressive tax, the poor pay a higher rate than the rich.
    c) Everyone pays the same amount with a lump-sum tax.
       Everyone pays the same rate with a flat tax.
    d) Tax deductions are subtracted from earned income before tax is calculated.
       Tax credits are subtracted from tax owed after tax is calculated.

Section 4.1

1. 3 red marbles  3. 3 spades  5. 4.4%  7. c. margin of error will increase
9. no, the margin of error would be too large.  11. 2500 people

Section 4.2

1. convenience  3. stratified  5. simple random  7. systematic
9. no blinding needed  11. Blinding
13. Are you satisfied with the variety offered in campus dining?
15. the same number may be rolled multiple times

Section 4.3

1. not valid, there are probably many meaningful three-digit sequences encountered during a single day.
3. valid, since she is considering five of any suit (not just spades) and still getting a rare result.
5. not valid, three consecutive wins by a player against an equal is not very rare (25%).

Section 4.4

3. 1970  5. just under half  7. pie chart  9. bar graph or frequency table
11. line chart

Section 4.5

1. strong negative correlation  3. no correlation  5. weak positive correlation
7. positive correlation  9. negative correlation  11. no correlation
13. age

Section 4.6

1. mode 2; median 2; mean 1.6  3. mode any; median 8404; mean 10,468.3
5. median  7. mean  9. 11

Section 4.7

1: a) 9  b) not surprised  c) surprised; the sample was biased
2: convenience sampling, no control group, leading question
3: a) There were 20 opportunities for this rare event to happen
    b) correlation does not imply cause
5: a) 12  b) 24  c) 22  d) 44%
6: a) mode 3.1; median 3.0; mean 3.01; mean is best
    b) mode any; median 86; mean 79.0; mean is best
    c) (in thousands) mode 220; median 255; mean 654; median is best
9.3 References

Section 2.6
[2.1] sabr.org
[2.2] cbssports.com

Section 3.3

Section 3.5
[3.4] taxfoundation.org
[3.5] www.sale-tax.com
[3.6] revenue-pa.custhelp.com

Section 4.2
[4.3] www.chantix.com

Section 4.3

Section 4.6

Figures & Tables:
2.3.3 www.google.com
2.5.1 www.census.gov
2.6.1 www.un.org
2.6.3 www.bls.gov/cpi
2.6.5 www.fhfa.gov
2.7.1 www.census.gov
3.5.1 taxfoundation.org
4.4.2 www.littleleague.org/series/history
4.4.4 weather.unisys.com/hurricane
4.4.7 usatoday.com
4.4.9  www.api.org
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